Reduction of one-loop amplitudes at the integrand level—NLO QCD calculations

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**Introduction: LHC needs NLO**

- The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course).
- In the last years we have seen a remarkable progress in the theoretical description of multi-particle processes at tree-order, thanks to very efficient recursive algorithms.
- The current need of precision goes beyond tree order. At LHC, most analyses require at least next-to-leading order calculations (NLO).
The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course)

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The current need of precision goes beyond tree order. At LHC, most analyses require at least next-to-leading order calculations (NLO)

As a result, a big effort has been devoted by several groups to the problem of an efficient computation of one-loop corrections for multi-particle processes!
Wishlist Les Houches 2007

1. \( pp \rightarrow V V + \text{jet} \)
2. \( pp \rightarrow t\bar{t} b\bar{b} \)
3. \( pp \rightarrow t\bar{t} + 2 \text{ jets} \)
4. \( pp \rightarrow W W W \)
5. \( pp \rightarrow V V b\bar{b} \)
6. \( pp \rightarrow V V + 2 \text{ jets} \)
7. \( pp \rightarrow V + 3 \text{ jets} \)
8. \( pp \rightarrow t\bar{t} b\bar{b} \)
9. \( pp \rightarrow 4 \text{ jets} \)

Processes for which a NLO calculation is both desired and feasible

Will we “finish” in time for LHC?
What has been done? (2005-2007)

Some recent results → Cross Sections available

- $pp \rightarrow ZZZ \rightarrow t\bar{t}Z$ [Lazopoulos, Melnikov, Petriello]
- $pp \rightarrow H + 2$ jets [Campbell, et al., J. R. Andersen, et al.]
- $pp \rightarrow VV + 2$ jets via VBF [Bozzi, Jäger, Oleari, Zeppenfeld]

Mostly $2 \rightarrow 3$, very few $2 \rightarrow 4$ complete calculations.

- $e^+ e^- \rightarrow 4$ fermions [Denner, Dittmaier, Roth]
- $e^+ e^- \rightarrow HH\nu\bar{\nu}$ [GRACE group (Boudjema et al.)]

This is NOT a complete list
(A lot of work has been done at NLO → calculations & new methods)
NLO troubles

Problems arising in NLO calculations

- Large Number of Feynman diagrams
- Reduction to Scalar Integrals (or sets of known integrals)
- Numerical Instabilities (inverse Gram determinants, spurious phase-space singularities)
- Extraction of soft and collinear singularities (we need virtual and real corrections)
**Methods available**

- **Traditional** Method: Feynman Diagrams & Passarino-Veltman Reduction:
  - general applicability major achievements
  - but major problem: not designed @ amplitude level
Methods available

- **Traditional** Method: Feynman Diagrams & Passarino-Veltman Reduction:

- **Semi-Numerical** Approach (Algebraic/Partly Numerical – Improved traditional) → Reduction to set of well-known integrals

- **Numerical** Approach (Numerical/Partly Algebraic) → Compute tensor integrals numerically
  - Ellis, Giele, Glover, Zanderighi;
  - Binoth, Guillet, Heinrich, Schubert;
  - Denner, Dittmaier; Del Aguila, Pittau;
  - Ferroglia, Passera, Passarino, Uccirati;
  - Nagy, Soper; van Hameren, Vppinga, Weinzierl;
**Methods available**

- **Traditional** Method: Feynman Diagrams & Passarino-Veltman Reduction:

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- **Analytic** Approach (Twistor-inspired)
  → extract information from lower-loop, lower-point amplitudes
  → determine scattering amplitudes by their poles and cuts
  - major advantage: designed to work @ amplitude level
  - **quadruple and triple cuts major simplifications**
  - Bern, Dixon, Dunbar, Kosower, Berger, Forde;
  - Anastasiou, Britto, Cachazo, Feng, Kunszt, Mastrolia;
Methods available

- **Traditional** Method: Feynman Diagrams & Passarino-Veltman Reduction:

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  → extract information from lower-loop, lower-point amplitudes
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☆ **OPP Integrand-level reduction combine:** PV@integrand + n-particle cuts
Any $m$-point one-loop amplitude can be written, before integration, as

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

A bar denotes objects living in $n = 4 + \epsilon$ dimensions

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2$$

$$\bar{q}^2 = q^2 + \tilde{q}^2$$

$$\bar{D}_i = D_i + \tilde{q}^2$$

External momenta $p_i$ are 4-dimensional objects
The old “master” formula

\[
\int A = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)
+ \sum_{i_0 < i_1 < i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2)
+ \sum_{i_0 < i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1)
+ \sum_{i_0}^{m-1} a(i_0) A_0(i_0)
+ \text{rational terms}
\]
General expression for the 4-dim $N(q)$ at the integrand level in terms of $D_i$

\[
N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
+ \sum_{i_0 < i_1 < i_2}^{m-1} \left[ c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
+ \sum_{i_0 < i_1}^{m-1} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
+ \sum_{i_0}^{m-1} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\]
The quantities \( d(i_0i_1i_2i_3) \) are the coefficients of 4-point functions with denominators labeled by \( i_0, i_1, i_2, \) and \( i_3 \).

\( c(i_0i_1i_2), b(i_0i_1), a(i_0) \) are the coefficients of all possible 3-point, 2-point and 1-point functions, respectively.
\[ N(q) = \sum_{i_0 < i_1 < i_2 < i_3} \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3} D_i + \sum_{i_0 < i_1 < i_2} \left[ c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2} D_i \\
+ \sum_{i_0 < i_1} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1} D_i + \sum_{i_0} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0} D_i \]

The quantities \( \tilde{d}, \tilde{c}, \tilde{b}, \tilde{a} \) are the “spurious” terms

- They still depend on \( q \) (integration momentum)
- They should vanish upon integration

What is the explicit expression of the spurious term?
Express any $q$ in $N(q)$ as

$$q^\mu = -p_0^\mu + \sum_{i=1}^{4} G_i \ell_i^\mu, \quad \ell_i^2 = 0$$

$$k_1 = \ell_1 + \alpha_1 \ell_2, \quad k_2 = \ell_2 + \alpha_2 \ell_1, \quad k_i = p_i - p_0$$

$$\ell_3^\mu = <\ell_1|\gamma^\mu|\ell_2>, \quad \ell_4^\mu = <\ell_2|\gamma^\mu|\ell_1>$$

The coefficients $G_i$ either reconstruct denominators $D_i$

$\rightarrow$ They give rise to $d$, $c$, $b$, $a$ coefficients
Express any $q$ in $N(q)$ as

$$q_\mu = -p^\mu_0 + \sum_{i=1}^{4} G_i \ell^\mu_i , \ell_i^2 = 0$$

- The coefficients $G_i$ either reconstruct denominators $D_i$ or vanish upon integration
- They give rise to $d, c, b, a$ coefficients
- They form the spurious $\tilde{d}, \tilde{c}, \tilde{b}, \tilde{a}$ coefficients
\( \tilde{d}(q) \) term (only 1)

\[
\tilde{d}(q) = \tilde{d} \, T(q),
\]

where \( \tilde{d} \) is a constant (does not depend on \( q \))

\[
T(q) \equiv Tr[(\gamma + \rho_0)\gamma_1\gamma_2\gamma_3\gamma_5]
\]
Spurious Terms - II

- $\tilde{d}(q)$ term (only 1)

$$\tilde{d}(q) = \tilde{d} \, T(q),$$

where $\tilde{d}$ is a constant (does not depend on $q$)

$$T(q) \equiv Tr[(\hat{q} + \hat{p}_0)\ell_1\ell_2\ell_3\gamma_5]$$

- $\tilde{c}(q)$ terms (they are 6)

$$\tilde{c}(q) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j} [(q + p_0) \cdot \ell_3]^j + \tilde{c}_{2j} [(q + p_0) \cdot \ell_4]^j \right\}$$

In the renormalizable gauge, $j_{max} = 3$
\( \tilde{d}(q) \) term (only 1)

\[
\tilde{d}(q) = \tilde{d} \, T(q),
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where \( \tilde{d} \) is a constant (does not depend on \( q \))

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T(q) \equiv Tr[(\not{q} + \not{p}_0) \gamma_1 \gamma_2 \gamma_3 \gamma_5]
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\]

In the renormalizable gauge, \( j_{\text{max}} = 3 \)

\( \tilde{b}(q) \) and \( \tilde{a}(q) \) give rise to 8 and 4 terms, respectively
A simple example

\[ \int \frac{1}{D_0 D_1 D_2 D_3 D_4} \]
A simple example

\[
\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \\
1 = \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}
\]
A simple example

\[
\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}
\]

\[
\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}
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A simple example

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\[ 1 = \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} \]

\[ \int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} \]

\[ \int \frac{1}{D_0 D_1 D_2 D_3 D_4} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \]
A simple example

\[
\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}
\]

\[
1 = \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}
\]

\[
\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}
\]

\[
\int \frac{1}{D_0 D_1 D_2 D_3 D_4} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)
\]

\[
d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left( \frac{1}{D_{i_4}(q^+)} + \frac{1}{D_{i_4}(q^-)} \right)
\]
A simple example

\[
\int \frac{1}{D_0 D_1 D_2 D_3 D_4} \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}
\]

Melrose, Nuovo Cim. 40 (1965) 181

Now we know the form of the spurious terms:

\[
N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i 
+ \sum_{i_0 < i_1}^{m-1} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i 
+ \sum_{i_0 < i_1}^{m-1} \left[ c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i 
+ \sum_{i_0 < i_1}^{m-1} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\]

Our calculation is now reduced to an algebraic problem.
General strategy

Now we know the form of the spurious terms:

\[ N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} [d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)] \prod_{i \neq i_0, i_1, i_2, i_3} D_i + \sum_{i_0 < i_1 < i_2}^{m-1} [c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)] \prod_{i \neq i_0, i_1, i_2} D_i + \sum_{i_0 < i_1}^{m-1} [b(i_0 i_1) + \tilde{b}(q; i_0 i_1)] \prod_{i \neq i_0, i_1} D_i + \sum_{i_0}^{m-1} [a(i_0) + \tilde{a}(q; i_0)] \prod_{i \neq i_0} D_i \]

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating \( N(q) \) for a set of values of the integration momentum \( q \)
Now we know the form of the spurious terms:

\[
N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2} D_i + \sum_{i_0 < i_1 < i_2}^{m-1} \left[ c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2} D_i \\
+ \sum_{i_0 < i_1}^{m-1} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1} D_i + \sum_{i_0}^{m-1} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0} D_i
\]

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating \( N(q) \) for a set of values of the integration momentum \( q \)

There is a very good set of such points: Use values of \( q \) for which a set of denominators \( D_i \) vanish → The system becomes “triangular”: solve first for 4-point functions, then 3-point functions and so on
Example: 4-particles process

\[ N(q) = d + \tilde{d}(q) + \sum_{i=0}^{3} [c(i) + \tilde{c}(q; i)] D_i + \sum_{i_0 < i_1}^{3} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] D_{i_0} D_{i_1} \]

\[ + \sum_{i_0=0}^{3} [a(i_0) + \tilde{a}(q; i_0)] D_{i \neq i_0} D_{j \neq i_0} D_{k \neq i_0} \]

We look for a \( q \) of the form \( q^\mu = -p_0^\mu + x_i \ell_i^\mu \) such that

\[ D_0 = D_1 = D_2 = D_3 = 0 \]

→ we get a system of equations in \( x_i \) that has two solutions \( q_0^\pm \)
**Example: 4-particles process**

\[ N(q) = d + \tilde{d}(q) \]

Our “master formula” for \( q = q_0^\pm \) is:

\[ N(q_0^\pm) = [d + \tilde{d} T(q_0^\pm)] \]

→ solve to extract the coefficients \( d \) and \( \tilde{d} \)
\[ N(q) - d - \tilde{d}(q) = \sum_{i=0}^{3} [c(i) + \tilde{c}(q; i)] D_i + \sum_{i_0 < i_1}^{3} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] D_{i_0} D_{i_1} \]

\[ + \sum_{i_0 = 0}^{3} \left[ a(i_0) + \tilde{a}(q; i_0) \right] D_{i \neq i_0} D_{j \neq i_0} D_{k \neq i_0} \]

Then we can move to the extraction of \( c \) coefficients using

\[ N'(q) = N(q) - d - \tilde{d} T(q) \]

and setting to zero three denominators (ex: \( D_1 = 0, D_2 = 0, D_3 = 0 \))
\[ N(q) - d - \tilde{d}(q) = [c(0) + \tilde{c}(q; 0)] D_0 \]

We have infinite values of \( q \) for which

\[ D_1 = D_2 = D_3 = 0 \quad \text{and} \quad D_0 \neq 0 \]

→ Here we need 7 of them to determine \( c(0) \) and \( \tilde{c}(q; 0) \)
Rational Terms - I

Let's go back to the integrand

\[ A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \]

Insert the expression for \( N(q) \rightarrow \) we know all the coefficients

\[ N(q) = \sum_{i_0 < i_1 < i_2 < i_3} [d + \tilde{d}(q)] \prod_{i \neq i_0, i_1, i_2, i_3} D_i + \sum_{i_0 < i_1 < i_2} [c + \tilde{c}(q)] \prod_{i \neq i_0, i_1, i_2} D_i + \cdots \]

Finally rewrite all denominators using

\[ \frac{D_i}{\bar{D}_i} = \bar{Z}_i, \text{ with } \bar{Z}_i \equiv \left(1 - \frac{\bar{q}^2}{D_i}\right) \]
Rational Terms - II

\[ A(\bar{q}) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) \bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}}{\bar{D}_0 \bar{D}_1 \bar{D}_2 \bar{D}_3} \prod_{i \neq i_0, i_1, i_2, i_3} \bar{Z}_i \]

\[ + \sum_{i_0 < i_1 < i_2}^{m-1} \frac{c(i_0 i_1 i_2) \bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \prod_{i \neq i_0, i_1, i_2} \bar{Z}_i \]

\[ + \sum_{i_0 < i_1}^{m-1} \frac{b(i_0 i_1) \bar{D}_{i_0} \bar{D}_{i_1}}{\bar{D}_0 \bar{D}_1} \prod_{i \neq i_0, i_1} \bar{Z}_i \]

\[ + \sum_{i_0}^{m-1} \frac{a(i_0) \bar{D}_{i_0}}{\bar{D}_0} \prod_{i \neq i_0} \bar{Z}_i \]

The rational part is produced, after integrating over \( d^n q \), by the \( \bar{q}^2 \) dependence in \( \bar{Z}_i \)

\[ \bar{Z}_i \equiv \left( 1 - \frac{\bar{q}^2}{\bar{D}_i} \right) \]
The “Extra Integrals” are of the form

\[ I_s^{(n;2\ell)} = \int d^n q \, \tilde{q}^{2\ell} \, \frac{q_{\mu_1} \cdots q_{\mu_r}}{\bar{D}(k_0) \cdots \bar{D}(k_s)}, \]

where

\[ \bar{D}(k_i) \equiv (\bar{q} + k_i)^2 - m_i^2, \quad k_i = p_i - p_0 \]

These integrals:
- have dimensionality \( D = 2(1 + \ell - s) + r \)
- contribute only when \( D \geq 0 \), otherwise are of \( O(\epsilon) \)
Rational Terms - IV

Tensor reduction iteratively leads to rank $m$ $m$-point tensors with $1 \leq m \leq 5$, that are ultraviolet divergent when $m \leq 4$. For this reason, we introduced, the $d$-dimensional denominators $\bar{D}_i$, that differs by an amount $\bar{q}^2$ from their 4-dimensional counterparts

$$\bar{D}_i = D_i + \bar{q}^2.$$  \hspace{1cm} (1)

The result of this is a mismatch in the cancelation of the $d$-dimensional denominators with the 4-dimensional ones. The rational part of the amplitude, called $R_1$, comes from such a lack of cancelation.

A different source of Rational Terms, called $R_2$, can also be generated from the $\epsilon$-dimensional part of $N(q)$

$R_2$ are generated by the difference

$$\bar{N}(\bar{q}) - N(q) = \bar{q}^2 N_1 + \epsilon N_2$$
\[
\bar{q}^\mu = q^\mu + \tilde{q}^\mu \\
\bar{\gamma}^\mu = \gamma^\mu + \tilde{\gamma}^\mu \\
\tilde{q}^\mu \rightarrow \tilde{q}^2 \\
\tilde{\gamma}^\mu \rightarrow \epsilon
\]
Rational Terms - IV

\[
\int d^n \tilde{q} \frac{\tilde{q}^2}{\tilde{D}_i \tilde{D}_j} = -\frac{i\pi^2}{2} \left[ m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3} \right] + O(\epsilon),
\]

\[
\int d^n \tilde{q} \frac{\tilde{q}^2}{\tilde{D}_i \tilde{D}_j \tilde{D}_k} = -\frac{i\pi^2}{2} + O(\epsilon), \quad \int d^n \tilde{q} \frac{\tilde{q}^4}{\tilde{D}_i \tilde{D}_j \tilde{D}_k \tilde{D}_l} = -\frac{i\pi^2}{6} + O(\epsilon) (2)
\]

\[
b(ij; \tilde{q}^2) = b(ij) + \tilde{q}^2 b^{(2)}(ij), \quad c(ijk; \tilde{q}^2) = c(ijk) + \tilde{q}^2 c^{(2)}(ijk). \quad (3)
\]

Furthermore, by defining

\[
\mathcal{D}^{(m)}(q, \tilde{q}^2) \equiv \sum_{i_0 < i_1 < i_2 < i_3} \left[ d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3} \tilde{D}_i, \quad (4)
\]
the following expansion holds

\[ D^{(m)}(q, \tilde{q}^2) = \sum_{j=2}^{m} \tilde{q}^{(2j-4)} d^{(2j-4)}(q), \] (5)

where the last coefficient is independent on \( q \)

\[ d^{(2m-4)}(q) = d^{(2m-4)}. \] (6)

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of \( \tilde{q}^2 \), in order to determine \( b^{(2)}(ij) \), \( c^{(2)}(ijk) \) and \( d^{(2m-4)}. \)
Calculate $N(q)$
We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly.
Calculate $N(q)$

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Evaluate scalar integrals

- massive integrals $\rightarrow$ FF [G. J. van Oldenborgh]
- massless integrals $\rightarrow$ OneLOop [A. van Hameren]
What we gain

- PV:

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  - A solid way to get all rational terms
Properties of the master equation
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- Polynomial equation in $q$
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The $N \equiv N$ test

A tool to efficiently treat phase-space points with numerical instabilities
As an example we present 4-photon and 6-photon amplitudes (via fermionic loop of mass $m_f$)

Input parameters for the reduction:
- External momenta $p_i$
- Masses of propagators in the loop
- Polarization vectors
As an example we present 4-photon and 6-photon amplitudes (via fermionic loop of mass $m_f$)

Input parameters for the reduction:
- External momenta $p_i$ → in this example massless, i.e. $p_i^2 = 0$
- Masses of propagators in the loop → all equal to $m_f$
- Polarization vectors → various helicity configurations
\[
\frac{F_{++++}^f}{\alpha^2 Q_f^4} = -8
\]
\[
\frac{F^{f+++-+}}{\alpha^2 Q_f^4} = -8 + 8 \left(1 + \frac{2\hat{u}}{\hat{s}}\right) B_0(\hat{u}) + 8 \left(1 + \frac{2\hat{t}}{\hat{s}}\right) B_0(\hat{t}) \\
- 8 \left(\frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2}\right) [\hat{t} C_0(\hat{t}) + \hat{u} C_0(\hat{u})] \\
- 4 \left[\frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2}\right] D_0(\hat{t}, \hat{u})
\]

Massless four-photon amplitudes
\[
\frac{F^f_{++++}}{\alpha^2 Q_f^4} = -8 + 8 \left( 1 + \frac{2\hat{u}}{\hat{s}} \right) B_0(\hat{u}) + 8 \left( 1 + \frac{2\hat{t}}{\hat{s}} \right) B_0(\hat{t}) \\
- 8 \left( \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} - \frac{4m_f^2}{\hat{s}} \right) [\hat{t} C_0(\hat{t}) + \hat{u} C_0(\hat{u})] \\
- 4 \left[ 4m_f^4 - (2\hat{s}m_f^2 + \hat{t}\hat{u}) \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} + \frac{4m_f^2\hat{t}\hat{u}}{\hat{s}} \right] D_0(\hat{t}, \hat{u}) \\
+ 8m_f^2(\hat{s} - 2m_f^2)[D_0(\hat{s}, \hat{t}) + D_0(\hat{s}, \hat{u})]
\]
\[ \frac{F^f_{++++}}{\alpha^2 Q^4_f} = -8 + 8 \left( 1 + \frac{2\hat{u}}{\hat{s}} \right) B_0(\hat{u}) + 8 \left( 1 + \frac{2\hat{t}}{\hat{s}} \right) B_0(\hat{t}) 
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Massless case: $[+ + - - - -] \text{ and } [+ - - + + -]$

Plot presented by Nagy and Soper hep-ph/0610028
(also Binoth et al., hep-ph/0703311)
Massless case: \([+ + \quad - - - - -] \) and \([+ - - + + -]\)

Analogous plot produced with OPP reduction
Massless case: $[+++---]$ and $[+++--++]$

Same plot as before for a wider range of $\theta$.
Massless case: \([+ + - - - -]\) and \([+ + - - + -]\)

Same idea for a different set of external momenta
Six Photons with Massive Fermions

Massless result [Mahlon]
Six Photons with Massive Fermions

- Massless result [Mahlon]
- $m = 0.5$ GeV
Massless result [Mahlon]

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- $m = 4.5$ GeV
Six Photons with Massive Fermions

Massless result [Mahlon]

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- $m = 4.5 \text{ GeV}$
- $m = 12.0 \text{ GeV}$
Six Photons with Massive Fermions

- Massless result [Mahlon]
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- $m = 4.5$ GeV
- $m = 12.0$ GeV
- $m = 20.0$ GeV
\[ q \bar{q} \rightarrow ZZZ \] VIRTUAL CORRECTIONS


\[ \frac{1}{\epsilon^2} \text{ and } \frac{1}{\epsilon} \]

\[ \sigma^{NLO,\text{virt}}_{\text{div}} = -C_F \frac{\alpha_s}{\pi} \frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}} (s_{12})^{-\epsilon} \left( \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) \sigma^{LO} \]
$q\bar{q} \rightarrow ZZZ$ VIRTUAL CORRECTIONS

A still naive implementation
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- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices!
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Comparison with LMP

- Of course full agreement for the $1/\epsilon^2$ and $1/\epsilon$ terms
- An 'easy' agreement for all graphs with up to 4-point loop integrals
- A bit more work to uncover the differences in scalar function normalization that happen to show to order $\epsilon^2$ thus influence only 5-point loop integrals.
q\bar{q} \rightarrow ZZZ  VIRTUAL CORRECTIONS
$q\bar{q} \rightarrow ZZZ$ VIRTUAL CORRECTIONS
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Typical precision:
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  \[
  -26.45706742815552
  -26.457067428165503661018557937723426
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  \begin{aligned}
  &-26.45706742815552 \\
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  \end{aligned}
  \]

Typical time: $10^4$ times faster (for non-singular PS-points)
\[ \sigma_{q\bar{q}}^{NLO} = \int_{VVV} \left[ d\sigma_{q\bar{q}}^{B} + d\sigma_{q\bar{q}}^{V} + d\sigma_{q\bar{q}}^{C} + \int_{g} d\sigma_{q\bar{q}}^{A} \right] + \int_{VVVg} \left[ d\sigma_{q\bar{q}}^{R} - d\sigma_{q\bar{q}}^{A} \right] \]

\[ \sigma_{gq}^{NLO} = \int_{VVV} \left[ +d\sigma_{gq}^{C} \int_{g} d\sigma_{gq}^{A} \right] + \int_{VVVg} \left[ d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right] , \]
\[ \sigma^{NLO}_{q\bar{q}} = \int_{VVV} \left[ d\sigma^B_{q\bar{q}} + d\sigma^V_{q\bar{q}} + d\sigma^C_{q\bar{q}} + \int_g d\sigma^A_{q\bar{q}} \right] + \int_{VVVg} \left[ d\sigma^R_{\bar{q}q} - d\sigma^A_{q\bar{q}} \right] \]

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\[ \mathcal{D}^{q_1g_6,\bar{q}_2} = \frac{8\pi\alpha_s C_F}{2\tilde{x}} \frac{p_1 \cdot p_6}{p_1 \cdot p_2 - p_2 \cdot p_6 - p_1 \cdot p_6} \left( \frac{1 + \tilde{x}^2}{1 - \tilde{x}} \right) |\mathcal{M}^B_{q\bar{q}}(\{\tilde{p}\})|^2 \]

\[ \tilde{x} = \frac{p_1 \cdot p_2 - p_2 \cdot p_6 - p_1 \cdot p_6}{p_1 \cdot p_2} \]
\[ \sigma_{q\bar{q}}^{NLO} = \int_{VVV} \left[ d\sigma_{q\bar{q}}^B + d\sigma_{q\bar{q}}^V + d\sigma_{q\bar{q}}^C + \int_g d\sigma_{q\bar{q}}^A \right] + \int_{VVVg} \left[ d\sigma_{q\bar{q}}^R - d\sigma_{q\bar{q}}^A \right] \]

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\[ \tilde{x} = \frac{p_1 \cdot p_2 - p_2 \cdot p_6 - p_1 \cdot p_6}{p_1 \cdot p_2} \]

\[ d\sigma_{q\bar{q}}^R - d\sigma_{q\bar{q}}^A = \frac{1}{6} \frac{1}{N} \frac{1}{2s_{12}} \left[ C_F |\mathcal{M}_{q\bar{q}}^R(\{p_j\})|^2 - \mathcal{D}_{q_1g_6,\bar{q}_2} - \mathcal{D}_{\bar{q}_2g_6,q_1} \right] d\Phi_{VVVg} \]
$$d\sigma^C_{q\bar{q}} + \int_g d\sigma^A_{q\bar{q}} = \frac{\alpha_s C_F}{2\pi} \frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}} \left( \frac{s_{12}}{\mu^2} \right)^{-\epsilon} \left[ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2\pi^2}{3} \right] d\sigma_B(\{p_j\})$$

$$+ \frac{\alpha_s C_F}{2\pi} \int_0^1 dx \ K_{q\bar{q}}(x) \ d\sigma_B(x p_1, p_2; p_3, p_4, p_5) \ F_0(x p_1, p_2; p_3, p_4, p_5)$$

$$+ \frac{\alpha_s C_F}{2\pi} \int_0^1 dx \ K_{q\bar{q}}(x) \ d\sigma_B(p_1, x p_2; p_3, p_4, p_5) \ F_0(p_1, x p_2; p_3, p_4, p_5)$$
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$$+ \frac{\alpha_s C_F}{2\pi} \int_0^1 dx \, K_{q\bar{q}}(x) \, d\sigma_B(xp_1, p_2; p_3, p_4, p_5) \, F_0(xp_1, p_2; p_3, p_4, p_5)$$

$$+ \frac{\alpha_s C_F}{2\pi} \int_0^1 dx \, K_{q\bar{q}}(x) \, d\sigma_B(p_1, xp_2; p_3, p_4, p_5) \, F_0(p_1, xp_2; p_3, p_4, p_5)$$

$$K_{q\bar{q}}(x) = \left( \frac{1+x^2}{1-x} \right) \log \left( \frac{s_{12}}{\mu^2_F} \right) + \left( \frac{4 \log(1-x)}{1-x} \right) + (1-x) - 2(1+x) \log(1-x)$$
\[ q\bar{q} \rightarrow ZZZ \quad \text{REAL CORRECTIONS} \]

\[ \sigma_{gq}^{NLO} = \int_{VVV} \left[ \int_g d\sigma^A_{gq} + d\sigma^C_{gq} \right] + \int_{VVVg} \left[ d\sigma^R_{gq} - d\sigma^A_{gq} \right] \]
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\[ \sigma_{gq}^{NLO} = \int_{VVV} \left[ \int_{g} d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int_{VVVg} \left[ d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right] \]

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\[ D_{g_{1}q_{6},q_{2}}^{g_{1}q_{6},q_{2}} = \frac{8\pi\alpha_{s} T_{R}}{\tilde{x} 2 p_{1} \cdot p_{6}} \left[ 1 - 2 \tilde{x} (1 - \tilde{x}) \right] |M_{q\bar{q}}^{B}(\{\tilde{p}_{j}\})|^{2} \]
\[ \frac{d\sigma^C_{gq}}{d\bar{q}} + \int d\sigma^A_{gq} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx \mathcal{K}_{gq}(x) d\sigma_B(xp_1, p_2; p_3, p_4, p_5) F_0(xp_1, p_2; p_3, p_4, p_5) \]

\[ \mathcal{K}_{gq}(x) = [x^2 + (1 - x)^2] \log \left( \frac{s_{12}}{\mu_F^2} \right) + 2x(1 - x) + 2[x^2 + (1 - x)^2] \log(1 - x) \]
\[ d\sigma^C_{\bar{q}q} + \int d\sigma^A_{\bar{q}q} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx \mathcal{K}_{\bar{q}q}(x) d\sigma_B(xp_1, p_2; p_3, p_4, p_5) F_0(xp_1, p_2; p_3, p_4, p_5) \]

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- check also with phase-space slicing method
\[ q\bar{q} \rightarrow ZZZ \text{ NLO} \]

<table>
<thead>
<tr>
<th>scale</th>
<th>( \sigma_0 )</th>
<th>( \sigma_V/\sigma_0 )</th>
<th>( \sigma_R )</th>
<th>( \sigma_{NLO} )</th>
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<tbody>
<tr>
<td>( \mu = M_Z )</td>
<td>1.481(5)</td>
<td>0.536(1)</td>
<td>0.238(2)</td>
<td>2.512(2)</td>
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<tr>
<td>( \mu = 4M_Z )</td>
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<td>2.355(2)</td>
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<tr>
<td>( \mu = 5M_Z )</td>
<td>1.479(5)</td>
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</tbody>
</table>
$q\bar{q} \rightarrow ZZZ \text{ NLO}$
Reduction at the integrand level
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- changes the computational approach at one loop
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CUTTOOLS version 0. is ready!