Reducing full one-loop amplitudes at the integrand level

Costas Papadopoulos, Les Houches 2007

In collaboration with G. Ossola and R. Pittau

HEP-NCSR DEMOKRITOS

The History

- Passarino-Veltman reduction
  general applicability major achievements
  major problem: not designed @ amplitude level
- Unitarity based methods Bern Dixon Dunbar Kosower
  major advantage: designed to work @ amplitude level
  limited applications
  - New insight, quadruple and triple cuts Britto Cachazo Feng …
  major simplifications
- Reduction at the integrand level Ossola Papadopoulos Pittau
  combine: PV@integrand + n-particle cuts

The Concept

- The master equation

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}, \qquad \bar{D}_i = (\bar{q} + p_i)^2 - m_i^2, \qquad p_0 \neq 0,$$

$$q \to D_i$$

$$q \cdot p_i \to D_i - D_0 + f$$

The Concept

- The master equation

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} D_i \, .
\end{aligned}
$$

Nuclear Physics B 763 (2007) 147–169

The Concept

- The master equation

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \cancel{\tilde{d}(q; i_0 i_1 i_2 i_3)} \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \cancel{\tilde{c}(q; i_0 i_1 i_2)} \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \cancel{\tilde{b}(q; i_0 i_1)} \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a(i_0) + \cancel{\tilde{a}(q; i_0)} \right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} D_i \,.
\end{aligned}
$$

Nuclear Physics B 763 (2007) 147–169

The Concept

- The master equation

$$
\begin{aligned}
A = & \sum_{i_0<i_1<i_2<i_3} d(i_0 i_1 i_2 i_3) D(i_0 i_1 i_2 i_3) \\
+ & \sum_{i_0<i_1<i_2} c(i_0 i_1 i_2) C(i_0 i_1 i_2) \\
+ & \sum_{i_0<i_1} b(i_0 i_1) B(i_0 i_1) \\
+ & \sum_{i_0} a(i_0) A(i_0) \\
+ & \ \mathrm{Rational}
\end{aligned}
$$

The Concept

- A simple application

$$\frac{1}{D_0 D_1 D_2 D_3 D_4}$$

$$N(q) = 1 = \sum_{i=0}^{4} (d_i + \tilde{d}_i) D_i$$

The Concept

- A simple application

$$\int d^n q \frac{\sum_{i=0}^{4}(d_i + \tilde{d}_i) D_i}{D_0 D_1 D_2 D_3 D_4} = \sum_{i=0}^{4} d_i \int d^n q \frac{D_i}{D_0 D_1 D_2 D_3 D_4}$$

The Concept

- The scalar 5-point function

$$I^5 = \sum_{i=0}^{4} d_i I^4(i)$$

$$d_i = \frac{1}{2}\left(\frac{1}{D_i(q_{(i)}^{+})} + \frac{1}{D_i(q_{(i)}^{-})}\right)$$

The Concept

- The scalar 5-point function

$$\frac{1}{2}\left(\frac{1}{D_i(q^{+}_{(i)})}+\frac{1}{D_i(q^{-}_{(i)})}\right)=-\frac{\det_i(Y^{(5)})}{\det(Y^{(5)})}$$

$$\begin{vmatrix} 2D_0+Y_{00} & D_1-D_0+Y_{10}-Y_{00} & D_2-D_0+Y_{20}-Y_{00} & \dots & D_5-D_0+Y_{20}-Y_{00} \\ D_1-D_0+Y_{10}-Y_{00} & Y_{11}-Y_{10}-Y_{01}+Y_{00} & Y_{12}-Y_{10}-Y_{02}+Y_{00} & \dots & Y_{15}-Y_{10}-Y_{05}+Y_{00} \\ D_2-D_0+Y_{20}-Y_{00} & Y_{21}-Y_{20}-Y_{01}+Y_{00} & Y_{22}-Y_{20}-Y_{02}+Y_{00} & \dots & Y_{25}-Y_{20}-Y_{05}+Y_{00} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_5-D_0+Y_{20}-Y_{00} & Y_{51}-Y_{50}-Y_{01}+Y_{00} & Y_{52}-Y_{50}-Y_{02}+Y_{00} & \dots & Y_{55}-Y_{50}-Y_{05}+Y_{00} \end{vmatrix}=0$$

- Imagine to calculate 10-point scalar integrals !

The Concept

- The master equation: a conceptual step

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} D_i \,.
\end{aligned}
$$

The Concept

- What we gain

  - PV: $N(q)$ or $A(q)$ hasn't to be known analytically

    No computer algebra

    Mathematica $\rightarrow$ Numerica

  - UM: more transparent algebraic method

    Rational terms

The Concept

- Algebraic no integration

Since the scalar 1-, 2-, 3-, 4-point functions are known, the only knowledge of the existence of the decomposition of Eq. (1.2) allows one to reduce the problem of calculating $A(\bar{q})$ to the algebraical problem of extracting all possible coefficients in Eq. (1.2) by computing $N(q)$ a sufficient number of times, at different values of q, and then inverting the system.

- Easily calculable

Notice that the described procedure can be performed *at the amplitude level*. One does not need to repeat the work for all Feynman diagrams, provided their sum is known. This circumstance is particularly appealing when our method is used together with some recursion relation to build up $N(q)$. We postpone this problem to a future publication and, in this paper, we suppose to know $N(q)$.

The Concept

- The master equation

  - Polynomial equation
  - Highly redundant: the a-terms have a degree of m^2-2 compared to m
  - Zeros of (a tower of) polynomial equations

    $$N(q)\text{-}d\text{-}terms\ etc$$

  - Different ways of solving it

The Concept

- Solving the master equation

- Choose a vector basis to expand the loop momentum
- Solve the propagator equations and find the loop momentum in terms of the external momenta involved
- Compute the ‘numerator’ at the given points

The Concept

- The master equation: ‘quadruple cut’

$$D_0 = D_1 = D_2 = D_3 = 0$$

$$N(q_0^{\pm}) = [d(0123) + \tilde{d}(q_0^{\pm}; 0123)] \prod_{i \neq 0,1,2,3} D_i(q_0^{\pm})$$

The Concept

- The ‘quadruple cut’

$$k_1 = \ell_1 + \alpha_1 \ell_2, \qquad k_2 = \ell_2 + \alpha_2 \ell_1$$

$$\ell_3^\mu = \langle \ell_1 | \gamma^\mu | \ell_2], \qquad \ell_4^\mu = \langle \ell_2 | \gamma^\mu | \ell_1]$$

$$q^\mu = -p_0^\mu + \frac{\beta}{\gamma} F^\mu - \frac{1}{2\gamma} Q^\mu + \sum_{i=0}^{2} \mathcal{O}(\bar{D}_i)$$

$$d(0123) + \tilde{d}(0123) T(q) + \sum_{i=0}^{3} \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2)$$

The Concept

- The spurious terms

$$\tilde{d}(q;0123) = \tilde{d}(0123)T(q)$$

$$T(q) \equiv \mathrm{Tr}\left[(\not{q} + \not{p}_0)\not{\ell}_1\not{\ell}_2\not{k}_3\gamma_5\right]$$

$$\int \frac{\mathrm{d}^4 Q\, \epsilon^{p_1 p_2 p_3 Q}}{N_0 N_1 N_2 N_3} = 0$$

W.L. van Neerven, J.A.M. Vermaseren **Phys.Lett.B137:241,1984**

The Concept

- The 'quadruple cut'

$$N(q_0^{\pm}) = [d(0123) + \tilde{d}(0123)T(q_0^{\pm})] \prod_{i \neq 0,1,2,3} D_i(q_0^{\pm})$$

$$R(q_0^{\pm}) \equiv \frac{N(q_0^{\pm})}{\prod_{i \neq 0,1,2,3} D_i(q_0^{\pm})}$$

The product of four tree-amplitudes

$$d(0123) = \frac{R(q_0^{-})T(q_0^{+}) - R(q_0^{+})T(q_0^{-})}{T(q_0^{+}) - T(q_0^{-})}$$

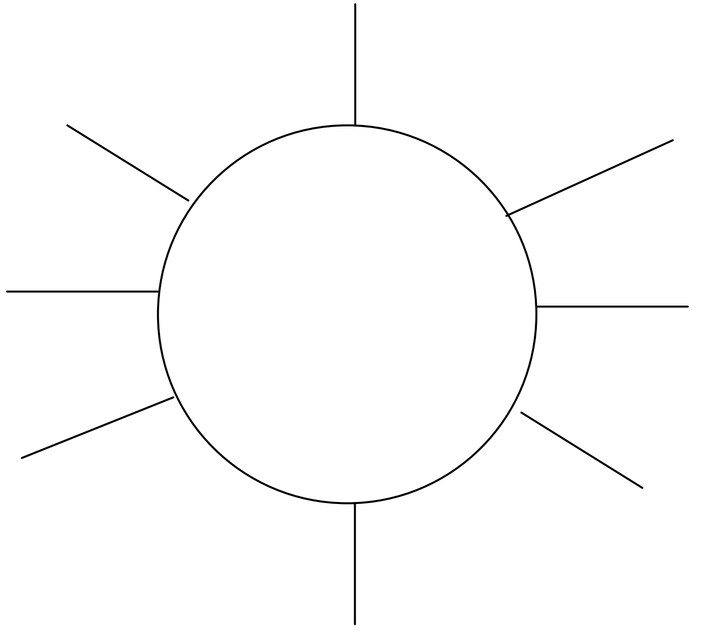
$$\tilde{d}(0123) = \frac{R(q_0^{+}) - R(q_0^{-})}{T(q_0^{+}) - T(q_0^{-})}.$$

The Concept

- The ‘quadruple cut’

$$R(q_0^\pm) \equiv \frac{N(q_0^\pm)}{\prod_{i\neq 0,1,2,3} D_i(q_0^\pm)}$$

The product of four tree-amplitudes

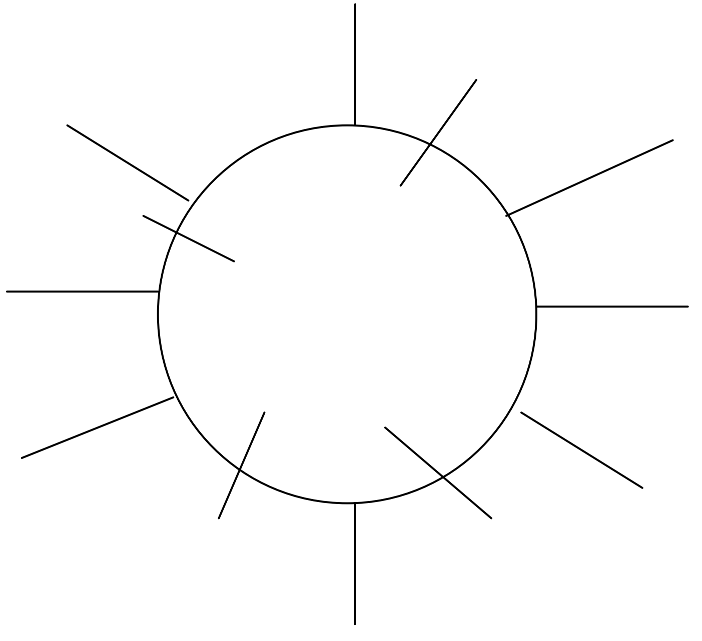


The Concept

- The ‘quadruple cut’

$$R(q_0^{\pm}) \equiv \frac{N(q_0^{\pm})}{\prod_{i \neq 0,1,2,3} D_i(q_0^{\pm})}$$

The product of four tree-amplitudes

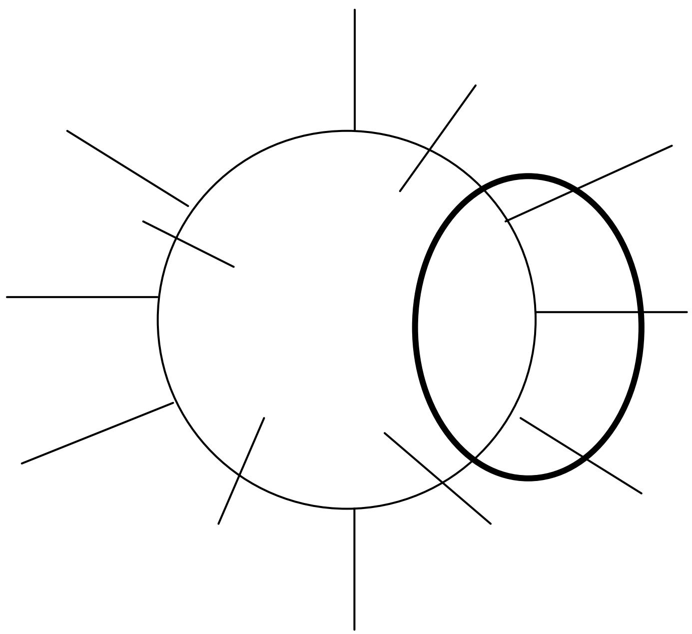


The Concept

- The ‘quadruple cut’

$$R(q_0^{\pm}) \equiv \frac{N(q_0^{\pm})}{\prod_{i\neq 0,1,2,3} D_i(q_0^{\pm})}$$

The product of four tree-amplitudes

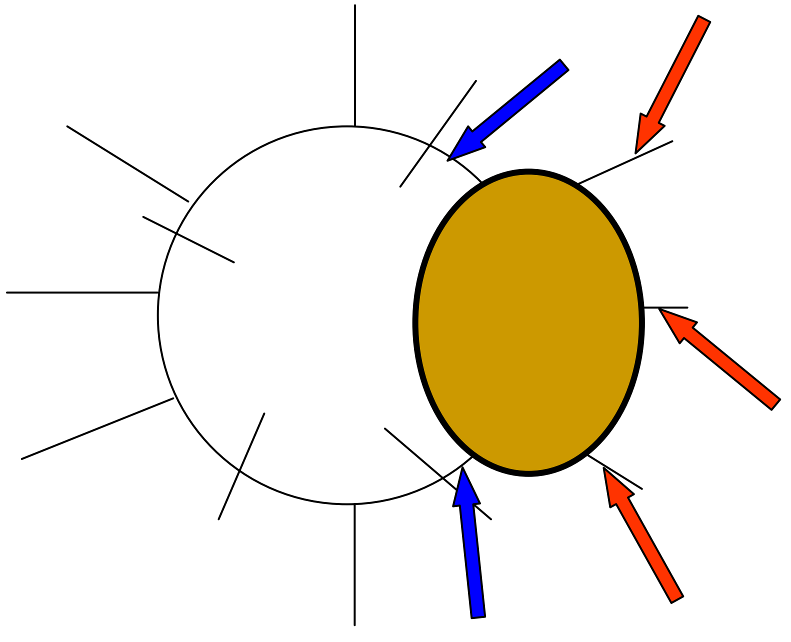


The Concept

- The 'quadruple cut'

$$R(q_0^{\pm}) \equiv \frac{N(q_0^{\pm})}{\prod_{i\neq 0,1,2,3} D_i(q_0^{\pm})}$$

The product of four tree-amplitudes



$$A_{tree}A_{tree}A_{tree}A_{tree}$$

In complex kinematics

The Concept

- The 'quadruple cut'

$$R(q_0^{\pm}) \equiv \frac{N(q_0^{\pm})}{\prod_{i\neq 0,1,2,3} D_i(q_0^{\pm})}$$

The product of four tree-amplitudes

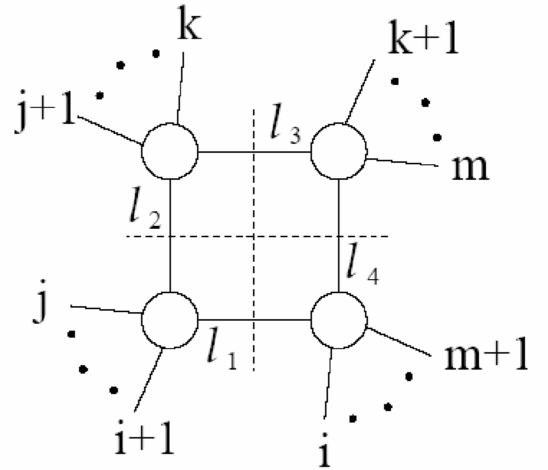


Fig. 2: A quadruple cut diagram. Momenta in the cut propagators flows clockwise and external momenta are taken outgoing. The tree-level amplitude $A_1^{\rm tree}$, for example, has external momenta $i+1, ..., j, \ell_2, \ell_1$.

The Concept

- The 'quadruple cut'

$$\int d\mu A^{\text{tree}}(\ell_1, i, \ldots, j, \ell_2) A^{\text{tree}}(-\ell_2, j+1, \ldots, i-1, -\ell_1) =$$
$$\sum \left(\widehat{b} \Delta I^{1m} + \widehat{c} \Delta I^{2m\ e} + \widehat{d} \Delta I^{2m\ h} + \widehat{g} \Delta I^{3m} + \widehat{f} \Delta I^{4m} \right).$$

However, extracting the coefficients is not a simple task in general. The main reason is that several scalar box integrals share a given cut, and therefore their unknown coefficients enter in the equation at the same time.

$$\widehat{f} = \frac{1}{|\mathcal{S}|} \sum_{\mathcal{S},J} n_J (A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}})$$

Britto, Cachazo, Feng Nucl.Phys.B725:275-305,2005.

The Concept

- The master equation: simple algebraic

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} D_i \,.
\end{aligned}
$$

The Concept

- Computing the 3-point coefficients
  - The role of spurious terms

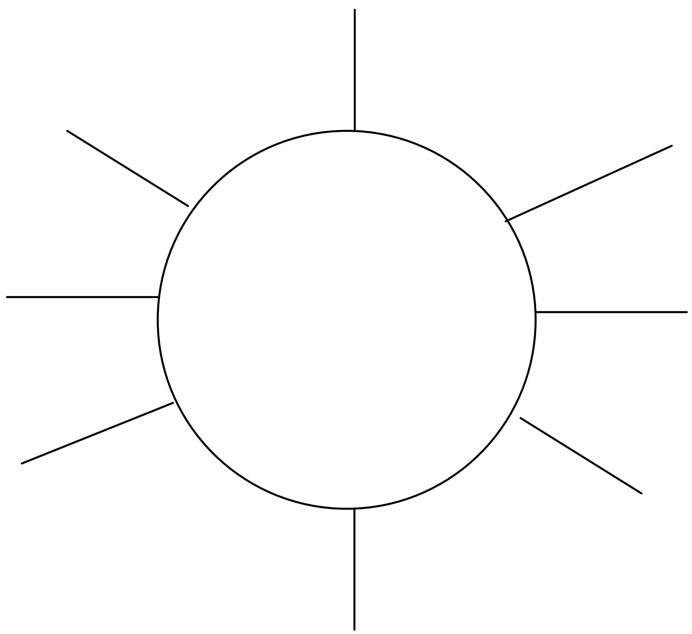
$$N(q) - \sum_{2<i_3} \left[d(012i_3) + \tilde{d}(q;012i_3) \right] \prod_{i \neq 0,1,2,i_3} D_i(q)$$

$$\equiv R'(q) \prod_{i \neq 0,1,2} D_i(q) = \left[c(012) + \tilde{c}(q;012) \right] \prod_{i \neq 0,1,2} D_i(q),$$

The Concept

- The ‘triple cut’

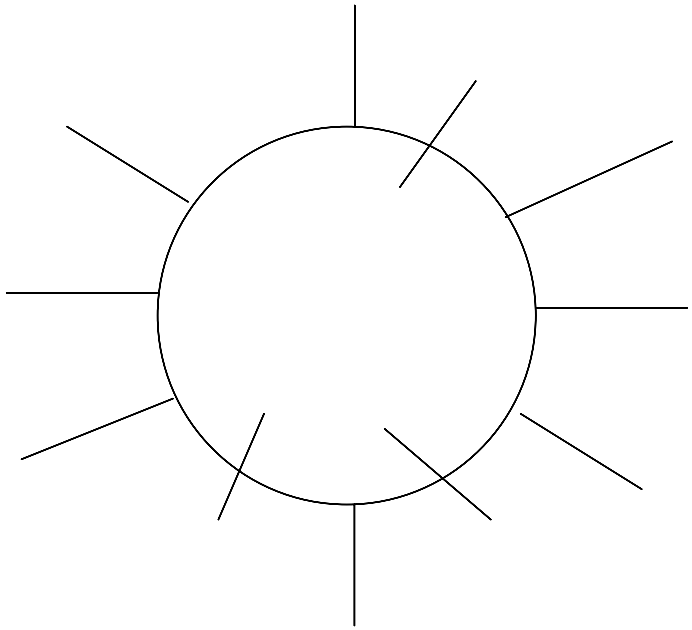
$$\frac{N(q)}{D_{i_0} D_{i_1} D_{i_2}}$$



The Concept

- The 'triple cut'

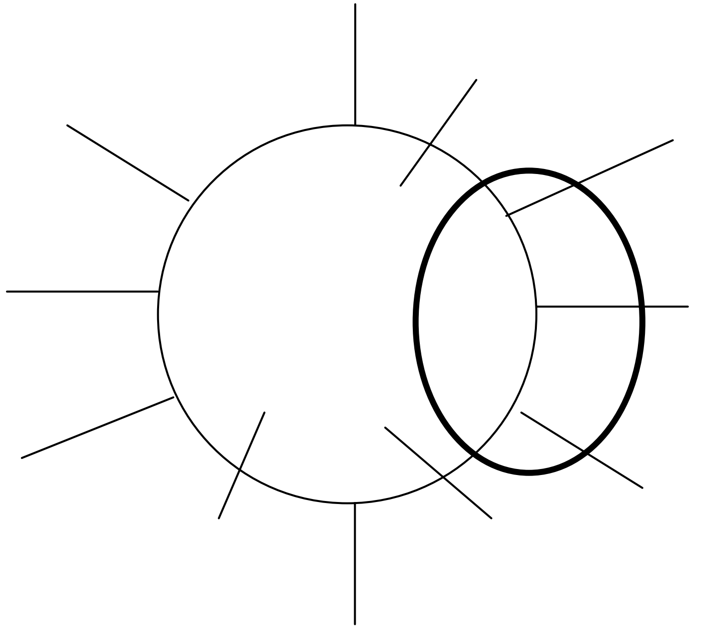
$$\frac{N(q)}{D_{i_0} D_{i_1} D_{i_2}}$$



The Concept

- The 'triple cut'

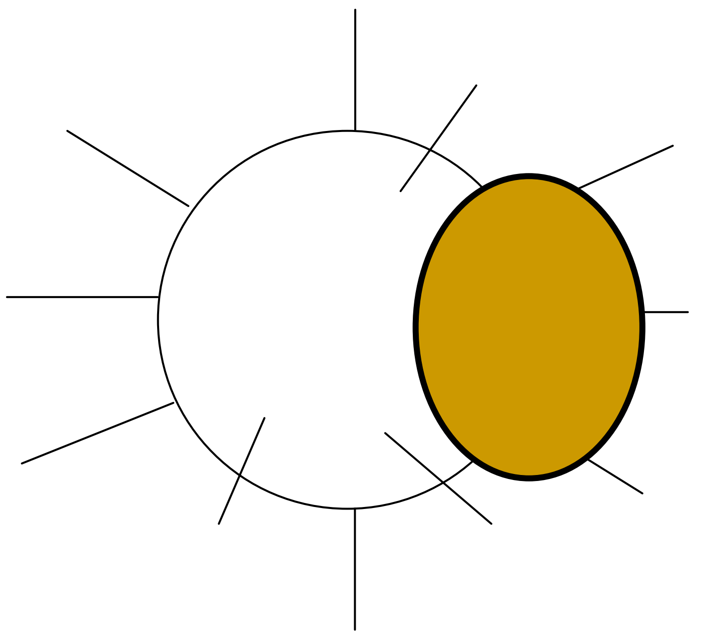
$$\frac{N(q)}{D_{i_0} D_{i_1} D_{i_2}}$$



The Concept

- The ‘triple cut’

$$\frac{N(q)}{D_{i_0} D_{i_1} D_{i_2}}$$



$$A_{tree} A_{tree} A_{tree}$$

In complex kinematics

The Concept

- The spurious terms

$$\tilde{c}(q;012) = \sum_{j=1}^{j_{\max}} \{\tilde{c}_{1j}(012)[(q+p_0)\cdot \ell_3]^j + \tilde{c}_{2j}(012)[(q+p_0)\cdot \ell_4]^j\}$$

$$j_{\max} = 3$$

We calculate 6 spurious terms. The solution is not unique - optimization

The Concept

- The 2-point sub-system and the stability

$$q^\mu = -p_0^\mu + y_1 k_1^\mu + y_n n^\mu + y_7 \ell_7^\mu + y_8 \ell_8^\mu .$$

$$n \cdot k_1 = 0 \quad \text{and} \quad n^2 = -k_1^2 .$$

$$q^\mu = -p_0^\mu + \frac{[(q+p_0)\cdot k_1]}{k_1^2} k_1^\mu - \frac{[(q+p_0)\cdot n]}{k_1^2} n^\mu + \frac{[(q+p_0)\cdot \ell_8]}{(\ell_7 \cdot \ell_8)} \ell_7^\mu + \frac{[(q+p_0)\cdot \ell_7]}{(\ell_7 \cdot \ell_8)} \ell_8^\mu .$$

Denner Dittmaier

The Concept

- The 2-point sub-system

$$N(q) - \sum_{1<i_2<i_3} \left[d(01i_2i_3) + \tilde{d}(q;01i_2i_3)\right] \prod_{i\neq 0,1,i_2,i_3} D_i$$

$$- \sum_{1<i_2} \left[c(01i_2) + \tilde{c}(q;01i_2)\right] \prod_{i\neq 0,1,i_2} D_i$$

$$\equiv R''(q) \prod_{i\neq 0,1} D_i(q) = \left[b(01) + \tilde{b}(q;01)\right] \prod_{i\neq 0,1} D_i(q)$$

The Concept

- The 2-point sub-system

$$\begin{aligned}
\tilde{b}(q;01) = {} & \tilde{b}_{11}(01)\big[(q+p_0)\cdot\ell_7\big] + \tilde{b}_{21}(01)\big[(q+p_0)\cdot\ell_8\big] \\
& + \tilde{b}_{12}(01)\big[(q+p_0)\cdot\ell_7\big]^2 + \tilde{b}_{22}(01)\big[(q+p_0)\cdot\ell_8\big]^2 \\
& + \tilde{b}_0(01)\big[(q+p_0)\cdot n\big] + \tilde{b}_{00}(01)K(q;01) \\
& + \tilde{b}_{01}(01)\big[(q+p_0)\cdot\ell_7\big]\big[(q+p_0)\cdot n\big] \\
& + \tilde{b}_{02}(01)\big[(q+p_0)\cdot\ell_8\big]\big[(q+p_0)\cdot n\big], \quad \textbf{with}
\end{aligned}$$

$$K(q;01) = \left\{\big[(q+p_0)\cdot n\big]^2 - \frac{[(q+p_0)\cdot k_1]^2 - (q+p_0)^2 k_1^2}{3}\right\}$$

The Concept

- The 2-point sub-system

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1}$$

$$N(q) = [b(01) + \tilde{b}(q;01)] + [a(0) + \tilde{a}(q;0)]D_1 + [a(1) + \tilde{a}(q;1)]D_0$$

$$k_1^2 \to 0\,, \quad \text{but} \quad k_1^\mu \neq 0\,,$$
$$k_1^2 \to 0\,, \quad \text{because} \quad k_1^\mu = 0$$
$$q^\mu = -p_0^\mu + y k_1^\mu + y_v v^\mu + y_7 \ell_7^\mu + y_8 \ell_8^\mu$$

$$\int d^n q \frac{[(q+p_0)\cdot v]^j}{\bar{D}_0 \bar{D}_1} \quad \text{with} \quad j = 1, 2 \quad \text{and} \quad v^2 = 0$$

The Concept

- The spurious terms

$$
\begin{aligned}
N(q) = {} & b + \hat{b}_0[(q+p_0)\cdot v] + \hat{b}_{00}[(q+p_0)\cdot v]^2 + \tilde{b}_{11}[(q+p_0)\cdot \ell_7] + \tilde{b}_{21}[(q+p_0)\cdot \ell_8] \\
& + \tilde{b}_{12}[(q+p_0)\cdot \ell_7]^2 + \tilde{b}_{22}[(q+p_0)\cdot \ell_8]^2 \\
& + \tilde{b}_{01}[(q+p_0)\cdot \ell_7][(q+p_0)\cdot v] \\
& + \tilde{b}_{02}[(q+p_0)\cdot \ell_8][(q+p_0)\cdot v] + \mathcal{O}(D_0) + \mathcal{O}(D_1)\,. \qquad \text{(B.7)}
\end{aligned}
$$

- To cure inverse determinants problem one needs to replace scalar integrals

We calculate 2+6 spurious terms. The solution is not unique - optimization

The Concept

- The case $k_1^\mu \to 0$

$$D(k_0) = D(k_1) + f + \mathcal{O}(k_1)$$

$$1 \equiv \frac{\bar{D}(k_0) - \bar{D}(k_1)}{f} + \frac{2(q \cdot k_1)}{f}$$

$$A(\bar{q}) = A^{(1)}(\bar{q}) + A^{(2)}(\bar{q}) + \mathcal{O}(k_1)\,,$$

$$A^{(1)}(\bar{q}) = \frac{1}{f}\frac{N(q)}{\bar{D}(k_1)}\,, \quad A^{(2)}(\bar{q}) = -\frac{1}{f}\frac{N(q)}{\bar{D}(k_0)}$$

  - Also $f \to 0$

The Concept

- The ‘rational terms’

$$\bar{D}_i = D_i + \tilde{q}^2 \qquad \frac{1}{\bar{D}_i} = \frac{\bar{Z}_i}{D_i}\,, \quad \text{with} \quad \bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)$$

$$A(\bar{q}) = \frac{N(q)}{D_0 D_1 \cdots D_{m-1}} \bar{Z}_0 \bar{Z}_1 \cdots \bar{Z}_{m-1}$$

The Concept

- The 'rational terms'

e-Print: **arXiv:0704.1271** [hep-ph]

$$
\begin{aligned}
A(\bar{q}) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{Z}_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{Z}_i \\
& + \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \prod_{i \neq i_0, i_1}^{m-1} \bar{Z}_i \\
& + \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \prod_{i \neq i_0}^{m-1} \bar{Z}_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} \bar{Z}_i \, .
\end{aligned}
$$

The Concept

- The ‘rational terms’

$$I^{(n;2\ell)}_{s;\mu_1\cdots\mu_r} \equiv \int d^n q\, \tilde{q}^{2\ell} \frac{q_{\mu_1}\cdots q_{\mu_r}}{\bar{D}(k_0)\cdots\bar{D}(k_s)}$$

$$\mathcal{D} = 2(1+\ell-s)+r \qquad \mathcal{D} \geq 0$$

$$\begin{aligned}
\tilde{P}(q) &= 0\,, \\
\tilde{a}(q;i_0) &= \tilde{a}^{\mu}(i_0;1)(q+p_{i_0})_\mu\,, \\
\tilde{b}(q;i_0i_1) &= \tilde{b}^{\mu}(i_0i_1;1)(q+p_{i_0})_\mu + \tilde{b}^{\mu\nu}(i_0i_1;2)(q+p_{i_0})_\mu(q+p_{i_0})_\nu\,, \\
\tilde{c}(q;i_0i_1i_2) &= \tilde{c}^{\mu}(i_0i_1i_2;1)(q+p_{i_0})_\mu + \tilde{c}^{\mu\nu}(i_0i_1i_1;2)(q+p_{i_0})_\mu(q+p_{i_0})_\nu\,, \\
&\quad + \tilde{c}^{\mu\nu\rho}(i_0i_1i_1;3)(q+p_{i_0})_\mu(q+p_{i_0})_\nu(q+p_{i_0})_\rho\,, \\
\tilde{d}(q;i_0i_1i_2i_3) &= \tilde{d}^{\mu}(i_0i_1i_2i_3;1)(q+p_{i_0})_\mu\,.
\end{aligned}$$

The Concept

- The 'rational terms'

Contributions proportional to $b(i_0 i_1)$

$$I_s^{(n;2(s-1))} = -i\pi^2 \frac{1}{s(s-1)} + \mathcal{O}(\epsilon)$$

Contributions proportional to $\tilde{b}^\mu(i_0 i_1; 1)$

$$I_{s;\mu}^{(n;2(s-1))} = i\pi^2 \frac{1}{(s+1)s(s-1)} \sum_{j=1}^{s} (k_j)_\mu + \mathcal{O}(\epsilon)$$

The Concept

- The 'rational terms'

Contributions proportional to $\tilde{b}^{\mu\nu}(i_0 i_1; 2)$

$$I_{s;\mu\nu}^{(n;2(s-1))} = -2i\pi^2 \frac{1}{(s+2)(s+1)s(s-1)} \left\{ \sum_{j=1}^{s} (k_j)_\mu (k_j)_\nu + \frac{1}{2} \sum_{j=1}^{s} \sum_{i\neq j}^{s} (k_j)_\mu (k_i)_\nu \right\}$$
$$+ \mathcal{O}(g_{\mu\nu}) + \mathcal{O}(\epsilon)\,. \tag{2.16}$$

Contributions proportional to $a(i_0)$

$$I_s^{(n;2s)} = -2i\pi^2 \frac{1}{(s+2)(s+1)s} \left\{ \sum_{j=1}^{s} k_j^2 + \frac{1}{2} \sum_{j=1}^{s} \sum_{i\neq j}^{s} (k_j \cdot k_i) + \frac{s+2}{2} \sum_{j=0}^{s} (m_j^2 - k_j^2) \right\}$$
$$+ \mathcal{O}(\epsilon)\,. \tag{2.17}$$

The Concept

- The 'rational terms'

Contributions proportional to $\tilde{a}^{\mu}(i_0;1)$

$$
\begin{aligned}
I_{s;\mu}^{(n;2s)} = i\pi^2 \frac{1}{(s+3)(s+2)(s+1)s} & \left\{ 6 \sum_{j=1}^{s} k_j^2 (k_j)_\mu + 2 \sum_{j=1}^{s} \sum_{i \neq j}^{s} \left[k_j^2 (k_i)_\mu + 2(k_j \cdot k_i)(k_j)_\mu \right] \right. \\
& + \sum_{j=1}^{s} \sum_{i \neq j}^{s} \sum_{\ell \neq i}^{s} (k_j \cdot k_i)(k_\ell)_\mu + (s+3) \left[2 \sum_{j=0}^{s} (m_j^2 - k_j^2)(k_j)_\mu \right. \\
& \left. \left. + \sum_{j=0}^{s} \sum_{i \neq j}^{s} (m_j^2 - k_j^2)(k_i)_\mu \right] \right\} + \mathcal{O}(\epsilon) \,. \qquad (2.18)
\end{aligned}
$$

The Applications

- photon amplitudes

$$\gamma + \gamma \rightarrow \gamma + \gamma$$

  - Calculated one loop amplitude before integration with Form or via spinor techniques
  - Found full agreement for both rational and non-rational terms, massive and massless

The Applications

- photon amplitudes

$$\gamma + \gamma \rightarrow \gamma + \gamma + \gamma + \gamma$$

  - Calculate one loop amplitude before integration with Form or via spinor techniques
  - Using both quadruple and double precision to cross check the results and to estimate numerical precision

The Applications

- photon amplitudes $\gamma + \gamma \rightarrow \gamma + \gamma + \gamma + \gamma$

Mahlon Nagy Soper Binoth Heinrich Gehrmann Mastrolia OPP

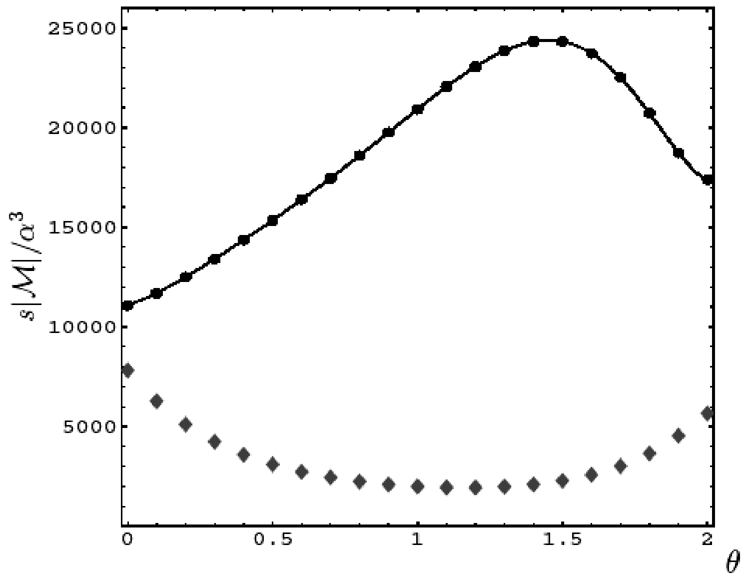


Figure 1: Comparison with Fig. 5 of Ref. [19]. Helicity configurations $[++----]$ and $[+--++-]$ for the momenta of Eq. (4.1), represented by black dots and gray diamonds respectively, and comparison with the analytic result of Ref. [21] (continuous line).

The Applications

- photon amplitudes $\gamma + \gamma \to \gamma + \gamma + \gamma + \gamma$

OPP

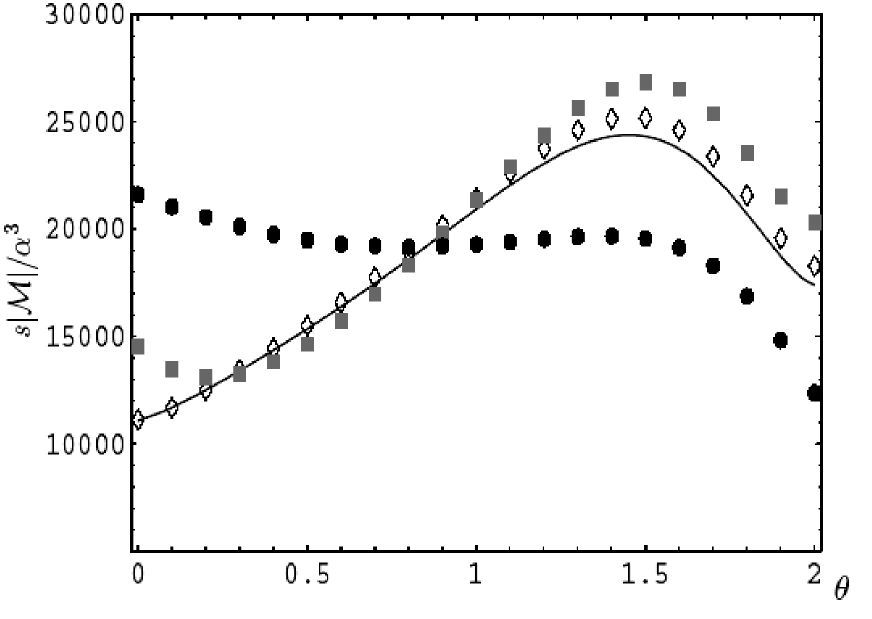


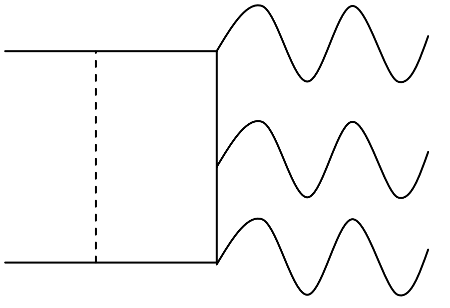
Figure 3: Helicity configuration $[++----]$ for the momenta of Eq. (4.1) for different values of the fermion mass in the loop: $m_f = 0.5$ GeV (diamond), $m_f = 4.5$ GeV (gray box) and $m_f = 12$ GeV (black dots). The continuous line is the result for the massless case.

The Applications

- QCD applications

$$pp \to ZZZ$$

$$q\bar{q} \to ZZZ$$



- All infrared poles in scalar functions

Lazopoulos Melnikov Petriello

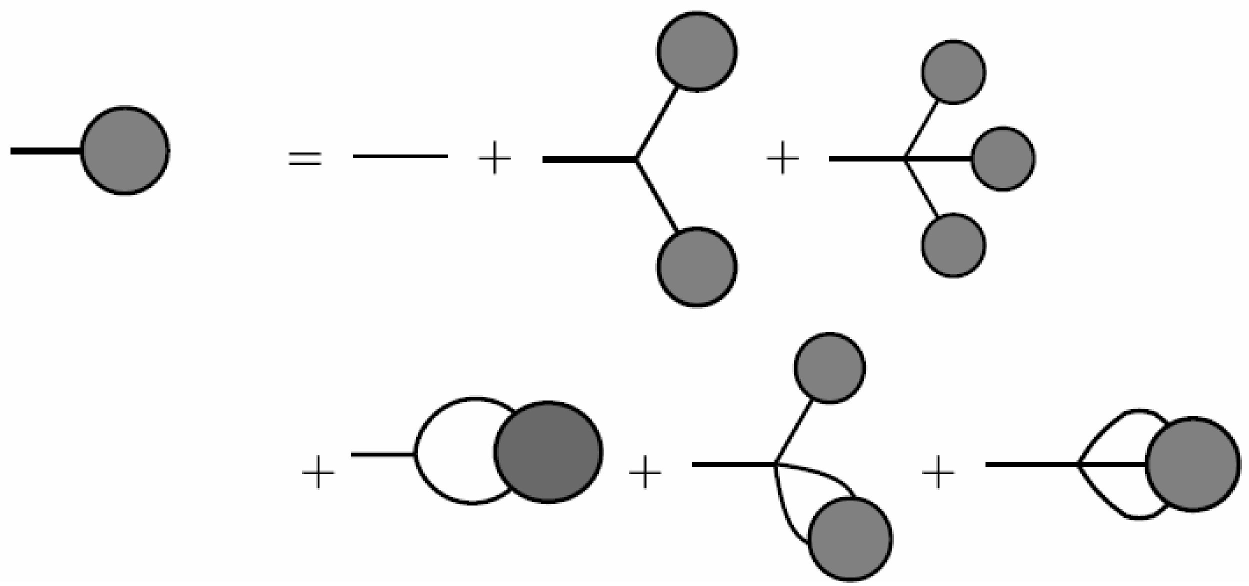
The Outlook

- Reduction at the integrand level
  - changes the computational approach at one loop
  - Numerical but still algebraic: speed and precision not a problem
- Future
  - Understand potential stability problems
  - Combine with the real corrections
  - Automatize through DS equations

A generic NLO calculator seems feasible

The Outlook

- The DS equations to all orders
  - Imagine a theory with 3- and 4- point vertices and just one field. Then it is straightforward to write an equation that gives the amplitude for $1 \to n$



The Outlook

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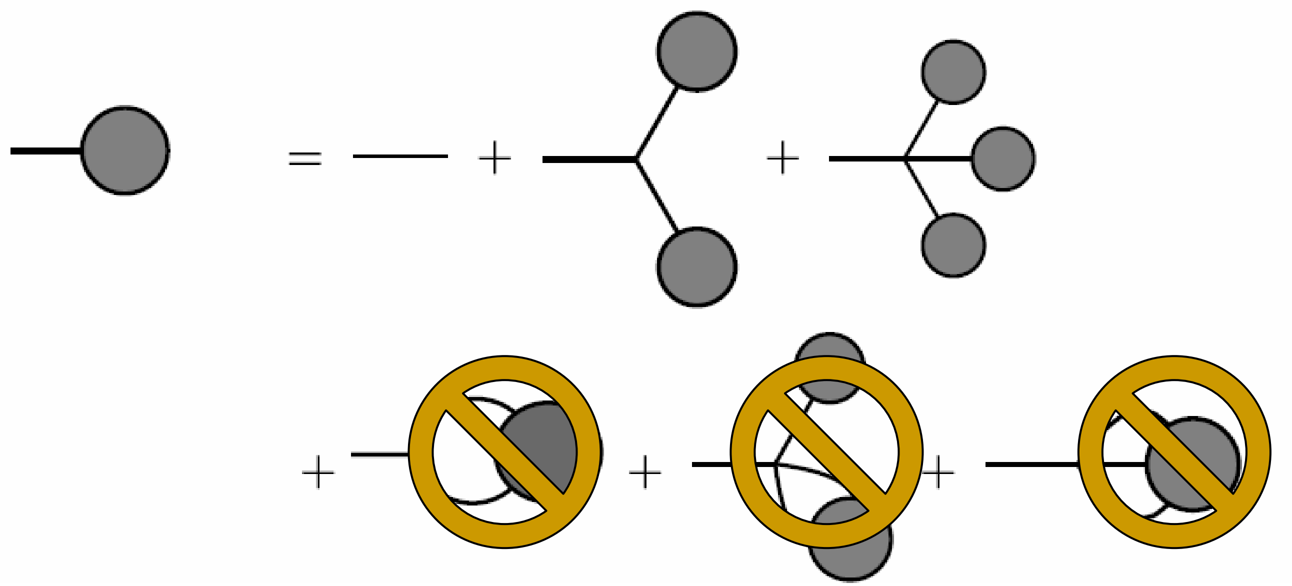
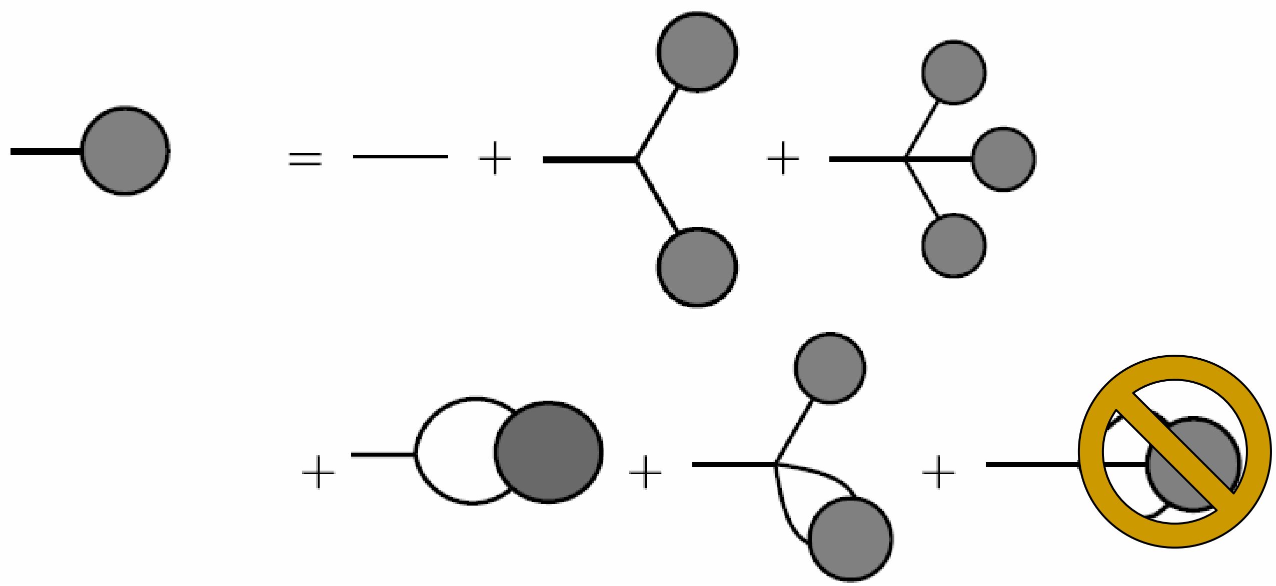
  - Imagine a theory with 3- and 4- point vertices and just one field. Then it is straightforward to write an equation that gives the amplitude for $1 \to n$



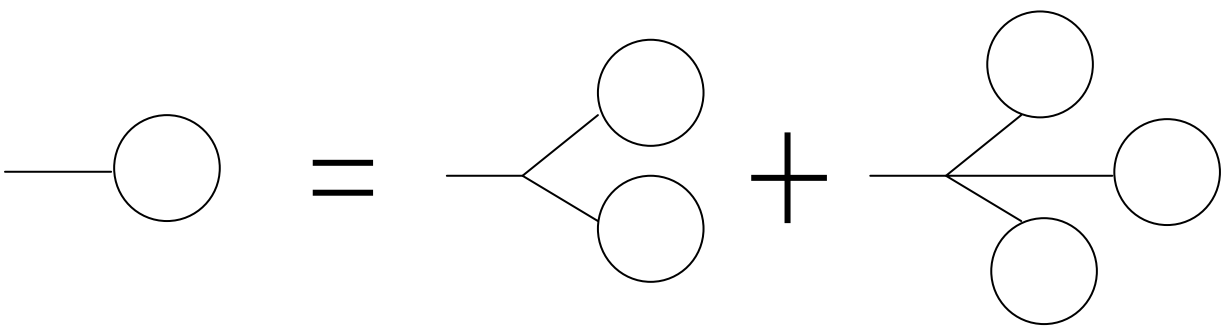
The Outlook

- The DS equations to all orders
  - Imagine a theory with 3- and 4- point vertices and just one field. Then it is straightforward to write an equation that gives the amplitude for $1 \rightarrow n$



The Outlook

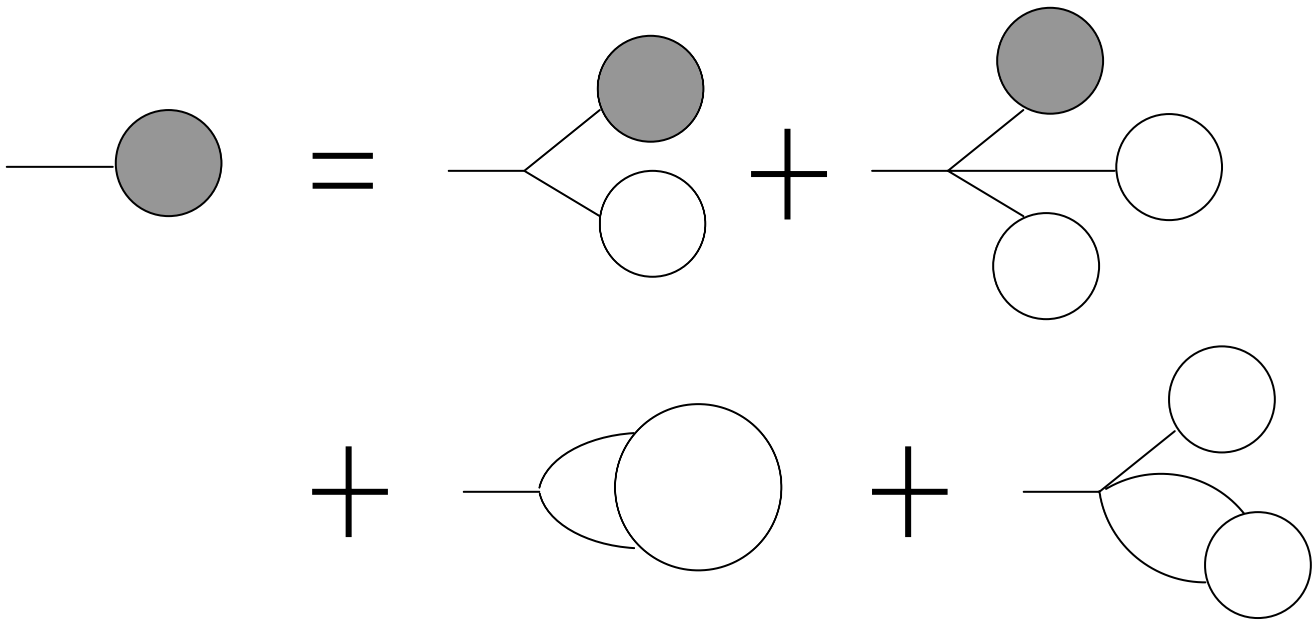
- The DS equations to tree order



- Combine 2,3,….,n external particles

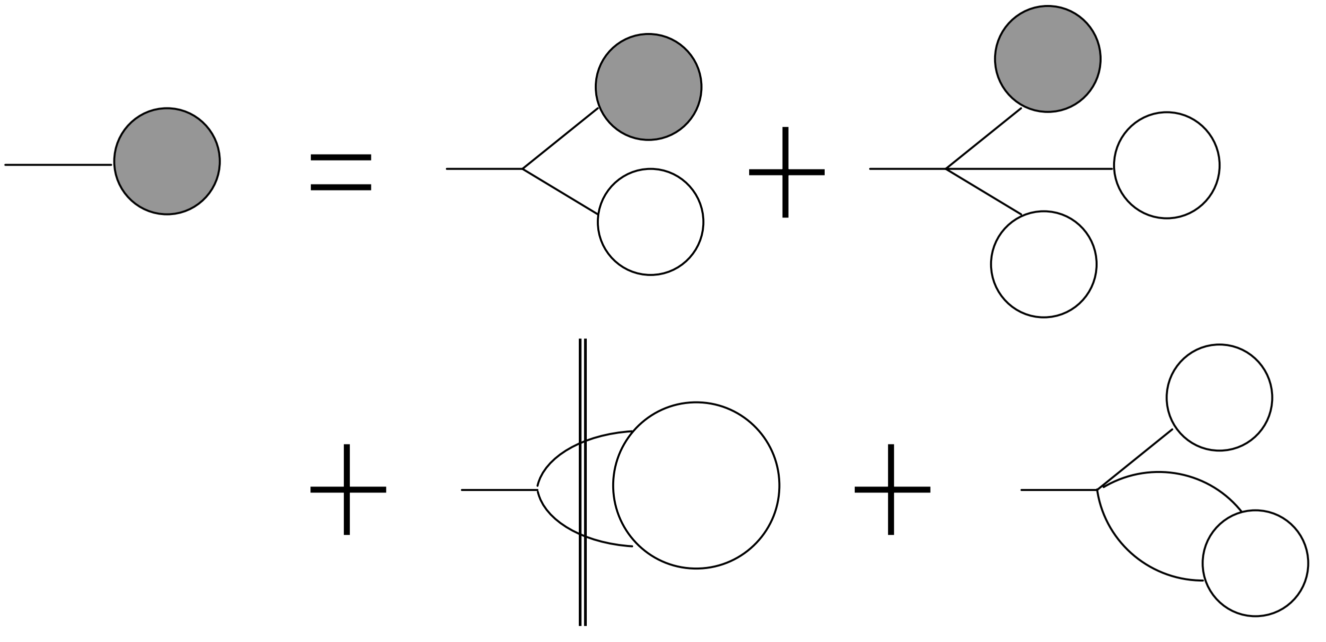
The Outlook

- The DS equations to one loop order linear !



The Outlook

- The DS equations to one loop order



- N+1 tree order sub-amplitudes