Reduction of one-loop amplitudes at the integrand level-NLO QCD calculations

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Outline

1. Introduction: Wishlists and Troubles
2. OPP Reduction
3. Numerical Tests
   - 4-photon amplitudes
   - 6-photon amplitudes
   - ZZZ production

Introduction: LHC needs NLO

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- The current need of precision goes beyond tree order. At LHC, most analyses require at least next-to-leading order calculations (NLO)
- As a result, a big effort has been devoted by several groups to the problem of an efficient computation of one-loop corrections for multi-particle processes!

NLO Wishlist Les Houches

[from G. Heinrich's Summary talk]

Wishlist Les Houches 2007
1. $pp \to V\,V + \text{jet}$
2. $pp \to t\bar{t}\,b\bar{b}$
3. $pp \to t\bar{t} + 2$ jets
4. $pp \to W\,W\,W$
5. $pp \to V\,V\,b\bar{b}$
6. $pp \to V\,V + 2$ jets
7. $pp \to V + 3$ jets
8. $pp \to t\bar{t}\,b\bar{b}$
9. $pp \to 4$ jets

Processes for which a NLO calculation is both desired and feasible

Will we "finish" in time for LHC?

What has been done? (2005-2007)

Some recent results $\rightarrow$ Cross Sections available

- $pp \rightarrow Z\,Z\,Z$ $pp \rightarrow t\bar{t}Z$ [Lazopoulos, Melnikov, Petriello]
- $pp \rightarrow H + 2$ jets [Campbell, et al., J. R. Andersen, et al.]
- $pp \rightarrow VV + 2$ jets via VBF [Bozzi, Jäger, Oleari, Zeppenfeld]

Mostly $2 \rightarrow 3$, very few $2 \rightarrow 4$ complete calculations.

- $e^+\ e^- \rightarrow 4$ fermions [Denner, Dittmaier, Roth]
- $e^+\ e^- \rightarrow H\,H\,\nu\,\bar{\nu}$ [GRACE group (Boudjema et al.)]

This is NOT a complete list

(A lot of work has been done at NLO $\rightarrow$ calculations & new methods)

NLO troubles

Problems arising in NLO calculations

- Large Number of Feynman diagrams
- Reduction to Scalar Integrals (or sets of known integrals)
- Numerical Instabilities (inverse Gram determinants, spurious phase-space singularities)
- Extraction of soft and collinear singularities (we need virtual and real corrections)

Methods available

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
  - general applicability major achievements
  - but major problem: not designed @ amplitude level

Methods available

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
- Semi-Numerical Approach (Algebraic/Partly Numerical – Improved traditional) $\rightarrow$ Reduction to set of well-known integrals
- Numerical Approach (Numerical/Partly Algebraic) $\rightarrow$ Compute tensor integrals numerically
  - Ellis, Giele, Glover, Zanderighi;
  - Binoth, Guillet, Heinrich, Schubert;
  - Denner, Dittmaier; Del Aguila, Pittau;
  - Ferroglia, Passera, Passarino, Uccirati;
  - Nagy, Soper; van Hameren, Vollinga, Weinzierl;

Methods available

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- Numerical Approach (Numerical/Partly Algebraic) $\rightarrow$ Compute tensor integrals numerically
- Analytic Approach (Twistor-inspired)
  $\rightarrow$ extract information from lower-loop, lower-point amplitudes
  $\rightarrow$ determine scattering amplitudes by their poles and cuts
  - major advantage: designed to work @ amplitude level
  - **quadruple and triple cuts major simplifications**
  - Bern, Dixon, Dunbar, Kosower, Berger, Forde;
  - Anastasiou, Britto, Cachazo, Feng, Kunszt, Mastrolia;

Methods available

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⋆ OPP Integrand-level reduction **combine: PV@integrand + n-particle cuts**

OPP Reduction - Intro

G. Ossola., C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007) – arXiv:hep-ph/0609007
and JHEP **0707** (2007) 085 – arXiv:0704.1271 [hep-ph]

Any m-point one-loop amplitude can be written, before integration, as

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

A bar denotes objects living in $n = 4 + \epsilon$ dimensions

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2$$

$$\bar{q}^2 = q^2 + \tilde{q}^2$$

$$\bar{D}_i = D_i + \tilde{q}^2$$

External momenta p_i are 4-dimensional objects

The old "master" formula

$$\int A = \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$
$$+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2)$$
$$+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1)$$
$$+ \sum_{i_0}^{m-1} a(i_0) A_0(i_0)$$
$$+ \text{ rational terms}$$

OPP "MASTER" FORMULA - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
+ & \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
+ & \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
+ & \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

OPP "MASTER" FORMULA - II

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) \qquad\qquad \right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c(i_0 i_1 i_2) \qquad\qquad] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) \qquad\qquad \right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1} [a(i_0) \qquad\qquad] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

- The quantities $d(i_0 i_1 i_2 i_3)$ are the coefficients of 4-point functions with denominators labeled by i_0, i_1, i_2, and i_3.
- $c(i_0 i_1 i_2)$, $b(i_0 i_1)$, $a(i_0)$ are the coefficients of all possible 3-point, 2-point and 1-point functions, respectively.

OPP "MASTER" FORMULA - II

$$
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N(q) \quad = & \quad \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
+ & \quad \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

The quantities $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ are the "spurious" terms

- They still depend on q (integration momentum)
- They should vanish upon integration

What is the explicit expression of the spurious term?

Spurious Terms - I

Following F. del Aguila and R. Pittau, arXiv:hep-ph/0404120

- Express any q in $N(q)$ as

$$q^\mu = -p_0^\mu + \sum_{i=1}^4 G_i\, \ell_i^\mu \;,\; {\ell_i}^2 = 0$$

$$k_1 = \ell_1 + \alpha_1 \ell_2\,, \quad k_2 = \ell_2 + \alpha_2 \ell_1\,,\; k_i = p_i - p_0$$
$${\ell_3}^\mu =< \ell_1|\gamma^\mu|\ell_2]\,,\; {\ell_4}^\mu =< \ell_2|\gamma^\mu|\ell_1]$$

- The coefficients G_i either reconstruct denominators D_i

$\rightarrow$ They give rise to d, c, b, a coefficients

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- The coefficients G_i either reconstruct denominators D_i or vanish upon integration

→ They give rise to d, c, b, a coefficients
→ They form the spurious $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ coefficients

- $\tilde{d}(q)$ term (only 1)

$$\tilde{d}(q) = \tilde{d}\, T(q)\,,$$

where $\tilde{d}$ is a constant (does not depend on q)

$$T(q) \equiv Tr[(\not{q} + \not{p}_0)\not{\ell}_1\not{\ell}_2\not{k}_3\gamma_5]$$

Spurious Terms - II

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- $\tilde{c}(q)$ terms (they are 6)

$$\tilde{c}(q) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j}[(q+p_0)\cdot\ell_3]^j + \tilde{c}_{2j}[(q+p_0)\cdot\ell_4]^j \right\}$$

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- $\tilde{b}(q)$ and $\tilde{a}(q)$ give rise to 8 and 4 terms, respectively

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

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- Melrose, Nuovo Cim. 40 (1965) 181
- G. Källén, J.Toll, J. Math. Phys. 6, 299 (1965)

General strategy

Now we know the form of the spurious terms:

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
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\end{aligned}
$$

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating N(q) for a set of values of the integration momentum q

There is a very good set of such points: Use values of q for which a set of denominators D_i vanish → The system becomes "triangular": solve first for 4-point functions, then 3-point functions and so on

Example: 4-particles process

$$
\begin{aligned}
N(q) &= d + \tilde{d}(q) + \sum_{i=0}^{3} \left[c(i) + \tilde{c}(q;i)\right] D_i + \sum_{i_0<i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q;i_0 i_1)\right] D_{i_0} D_{i_1} \\
&+ \sum_{i_0=0}^{3} \left[a(i_0) + \tilde{a}(q;i_0)\right] D_{i\neq i_0} D_{j\neq i_0} D_{k\neq i_0}
\end{aligned}
$$

We look for a q of the form $q^\mu = -p_0^\mu + x_i \ell_i^\mu$ such that

$$
D_0 = D_1 = D_2 = D_3 = 0
$$

$\rightarrow$ we get a system of equations in x_i that has two solutions $q_0^\pm$

Example: 4-particles process

$$N(q) \quad = \quad d + \tilde{d}(q)$$

Our "master formula" for $q = q_0^{\pm}$ is:

$$N(q_0^{\pm}) = [d + \tilde{d}\; T(q_0^{\pm})]$$

$\longrightarrow$ solve to extract the coefficients d and $\tilde{d}$

$$N(q) - d - \tilde{d}(q) = \sum_{i=0}^{3} \left[c(i) + \tilde{c}(q;i)\right] D_i + \sum_{i_0<i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] D_{i_0} D_{i_1}$$
$$+ \sum_{i_0=0}^{3} \left[a(i_0) + \tilde{a}(q; i_0)\right] D_{i\neq i_0} D_{j\neq i_0} D_{k\neq i_0}$$

Then we can move to the extraction of c coefficients using

$$N'(q) = N(q) - d - \tilde{d}\, T(q)$$

and setting to zero three denominators (ex: $D_1 = 0$, $D_2 = 0$, $D_3 = 0$)

Example: 4-particles process

$$N(q) - d - \tilde{d}(q) \quad = \qquad [c(0) + \tilde{c}(q;0)]\, D_0$$

We have infinite values of q for which

$$D_1 = D_2 = D_3 = 0 \quad \text{and} \quad D_0 \neq 0$$

$\longrightarrow$ Here we need 7 of them to determine $c(0)$ and $\tilde{c}(q;0)$

Rational Terms - I

- Let's go back to the integrand

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

- Insert the expression for $N(q) \rightarrow$ we know all the coefficients

$$N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d + \tilde{d}(q)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c + \tilde{c}(q)] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i + \cdots$$

- Finally rewrite all denominators using

$$\frac{D_i}{\bar{D}_i} = \bar{Z}_i \,, \qquad \text{with} \quad \bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)$$

Rational Terms - II

$$
\begin{aligned}
A(\bar{q}) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \prod_{i \neq i_0, i_1}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \prod_{i \neq i_0}^{m-1} \bar{Z}_i
\end{aligned}
$$

The rational part is produced, after integrating over $d^n q$, by the $\tilde{q}^2$ dependence in $\bar{Z}_i$

$$
\bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)
$$

Rational Terms - III

The "Extra Integrals" are of the form

$$I_{s;\mu_1\cdots\mu_r}^{(n;2\ell)} \equiv \int d^n q\, \tilde{q}^{2\ell} \frac{q_{\mu_1}\cdots q_{\mu_r}}{\bar{D}(k_0)\cdots\bar{D}(k_s)}\,,$$

where

$$\bar{D}(k_i) \equiv (\bar{q}+k_i)^2 - m_i^2\,,\, k_i = p_i - p_0$$

These integrals:

- **have dimensionality** $\mathcal{D} = 2(1+\ell-s)+r$
- **contribute only when** $\mathcal{D} \geq 0$**, otherwise are of** $\mathcal{O}(\epsilon)$

Rational Terms - IV

Tensor reduction iteratively leads to rank m m-point tensors with $1 \leq m \leq 5$, that are ultraviolet divergent when $m \leq 4$. For this reason, we introduced, the d-dimensional denominators $\bar{D}_i$, that differs by an amount $\tilde{q}^2$ from their 4-dimensional counterparts

$$\bar{D}_i = D_i + \tilde{q}^2 \,. \tag{1}$$

The result of this is a mismatch in the cancelation of the d-dimensional denominators with the 4-dimensional ones. The rational part of the amplitude, called R_1, comes from such a lack of cancelation.
A different source of Rational Terms, called R_2, can also be generated from the ϵ-dimensional part of $N(q)$
R_2 are generated by the difference

$$\bar{N}(\bar{q}) - N(q) = \tilde{q}^2 N_1 + \epsilon N_2$$

Rational Terms - IV

$$\bar{q}^\mu = q^\mu + \tilde{q}^\mu$$

$$\bar{\gamma}^\mu = \gamma^\mu + \tilde{\gamma}^\mu$$

$$\tilde{q}^\mu \to \tilde{q}^2$$

$$\tilde{\gamma}^\mu \to \epsilon$$

$$\int d^n\bar{q}\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} = -\frac{i\pi^2}{2}\left[m_i^2+m_j^2-\frac{(p_i-p_j)^2}{3}\right]+\mathcal{O}(\epsilon)\,,$$

$$\int d^n\bar{q}\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} = -\frac{i\pi^2}{2}+\mathcal{O}(\epsilon)\,,\quad \int d^n\bar{q}\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{6}+\mathcal{O}(\epsilon) \quad (2)$$

$$b(ij;\tilde{q}^2) = b(ij)+\tilde{q}^2 b^{(2)}(ij)\,,\quad c(ijk;\tilde{q}^2) = c(ijk)+\tilde{q}^2 c^{(2)}(ijk)\,. \quad (3)$$

Furthermore, by defining

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) \equiv \sum_{i_0<i_1<i_2<i_3}^{m-1}\left[d(i_0 i_1 i_2 i_3;\tilde{q}^2)+\tilde{d}(q;i_0 i_1 i_2 i_3;\tilde{q}^2)\right]\prod_{i\neq i_0,i_1,i_2,i_3}^{m-1}\bar{D}_i\,, \quad (4)$$

Rational Terms - IV

the following expansion holds

$$\mathcal{D}^{(m)}(q, \tilde{q}^2) = \sum_{j=2}^{m} \tilde{q}^{(2j-4)} d^{(2j-4)}(q)\,, \tag{5}$$

where the last coefficient is independent on q

$$d^{(2m-4)}(q) = d^{(2m-4)}\,. \tag{6}$$

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

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Evaluate scalar integrals

- massive integrals $\longrightarrow$ FF [G. J. van Oldenborgh]
- massless integrals $\longrightarrow$ OneLOop [A. van Hameren]

What we gain

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- Unitarity Methods:

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  - A solid way to get all rational terms

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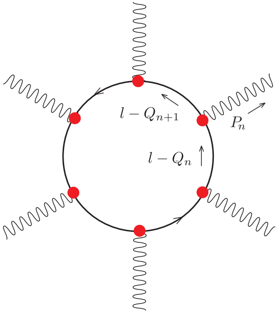
The $N \equiv N$ test

A tool to efficiently treat phase-space points with numerical instabilities

4-photon and 6-photon amplitudes

As an example we present 4-photon and 6-photon amplitudes (via fermionic loop of mass m_f)



Input parameters for the reduction:

- External momenta p_i
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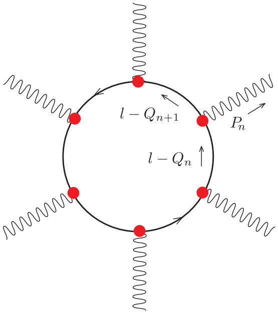

Input parameters for the reduction:

- External momenta p_i → in this example massless, i.e. $p_i^2 = 0$
- Masses of propagators in the loop → all equal to m_f
- Polarization vectors → various helicity configurations

$$\frac{F^f_{++++}}{\alpha^2 Q_f^4} = -8$$

Rational Part

$$
\begin{aligned}
\frac{F^f_{++++}}{\alpha^2 Q_f^4} \quad = \quad & -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right) B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right) B_0(\hat{t}) \\
& - 8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} \qquad\qquad \right)\left[\hat{t}\,C_0(\hat{t}) + \hat{u}\,C_0(\hat{u})\right] \\
& - 4\left[\qquad\qquad \frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} \qquad\qquad \right] D_0(\hat{t},\hat{u})
\end{aligned}
$$

Massless four-photon amplitudes

Four Photons – Comparison with *Gounaris et al.*

$$
\begin{aligned}
\frac{F^f_{++++}}{\alpha^2 Q_f^4} \;=\;& -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right)B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right)B_0(\hat{t}) \\
& - 8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} - \frac{4m_f^2}{\hat{s}}\right)[\hat{t}\,C_0(\hat{t}) + \hat{u}\,C_0(\hat{u})] \\
& - 4\left[4m_f^4 - (2\hat{s}m_f^2 + \hat{t}\hat{u})\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} + \frac{4m_f^2\hat{t}\hat{u}}{\hat{s}}\right]D_0(\hat{t},\hat{u}) \\
& + 8m_f^2(\hat{s}-2m_f^2)[D_0(\hat{s},\hat{t}) + D_0(\hat{s},\hat{u})]
\end{aligned}
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Massive four-photon amplitudes

$$
\begin{aligned}
\frac{F^f_{++++}}{\alpha^2 Q_f^4} &= -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right) B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right) B_0(\hat{t}) \\
&- 8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} - \frac{4m_f^2}{\hat{s}}\right)[\hat{t}C_0(\hat{t}) + \hat{u}C_0(\hat{u})] \\
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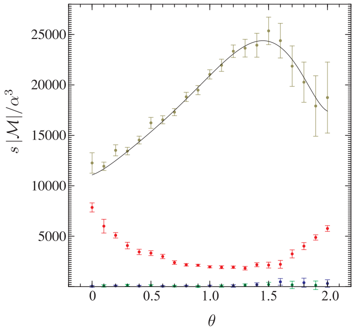
\end{aligned}
$$

Massive four-photon amplitudes

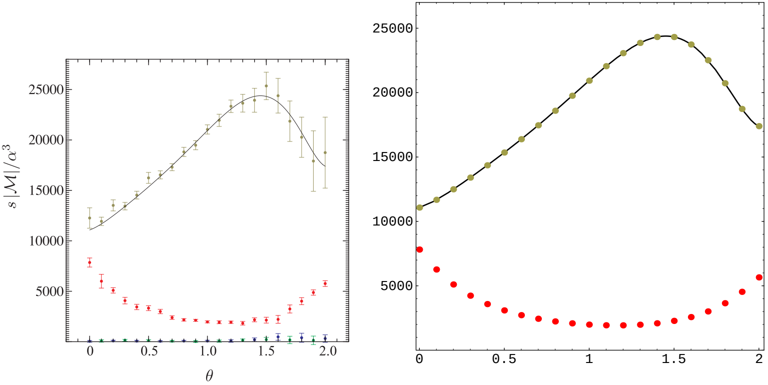
Results also checked for F^f_{+++-} and F^f_{++--}

Massless case: $[+ + - - - -]$ and $[+ - - + + -]$



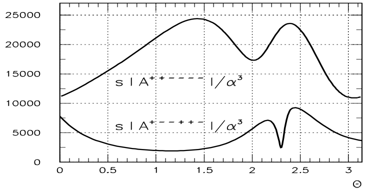
Plot presented by Nagy and Soper hep-ph/0610028
(also Binoth et al., hep-ph/0703311)

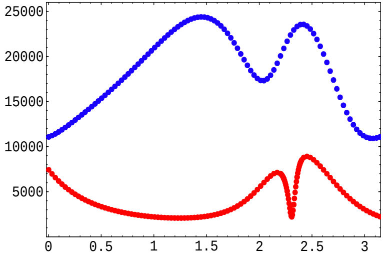
Massless case: $[++----]$ and $[+--++-]$



Analogous plot produced with OPP reduction

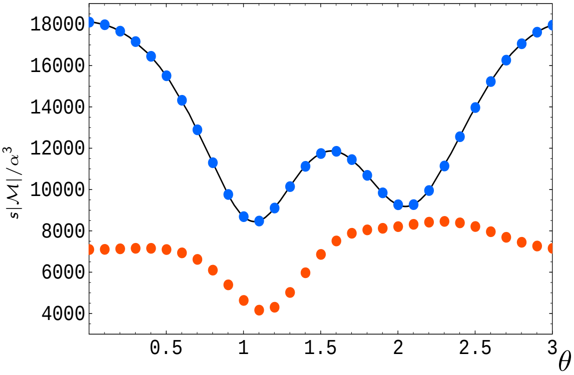
Massless case: $[++----]$ and $[++--+-]$





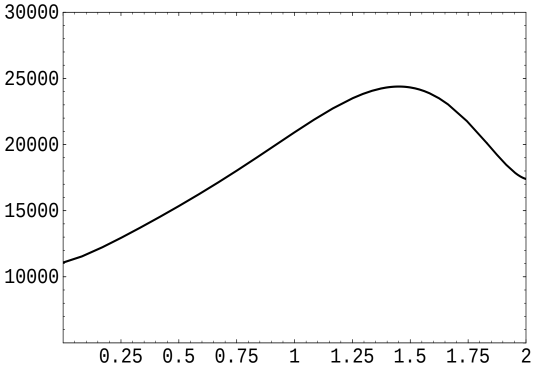
Same plot as before for a wider range of θ

Massless case: $[++----]$ and $[++--+-]$



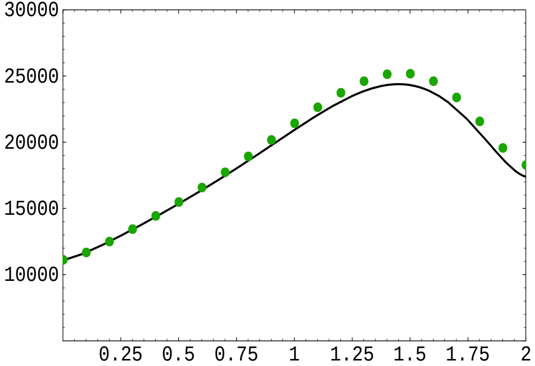
Same idea for a different set of external momenta

Six Photons with Massive Fermions



- Massless result [Mahlon]

Six Photons with Massive Fermions



- Massless result [Mahlon]
- $m = 0.5$ GeV

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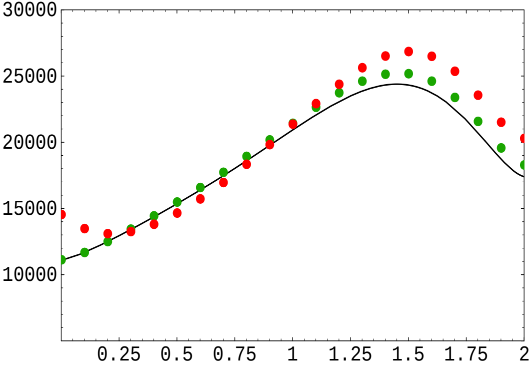

- Massless result [Mahlon]
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- $m = 4.5$ GeV

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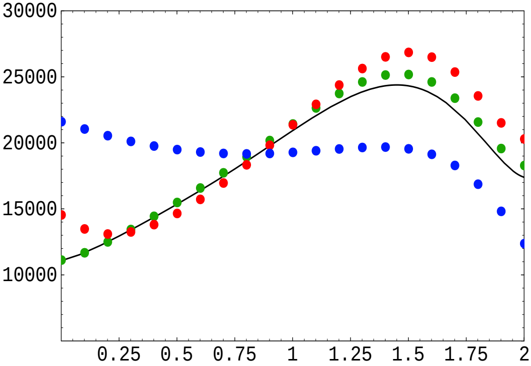

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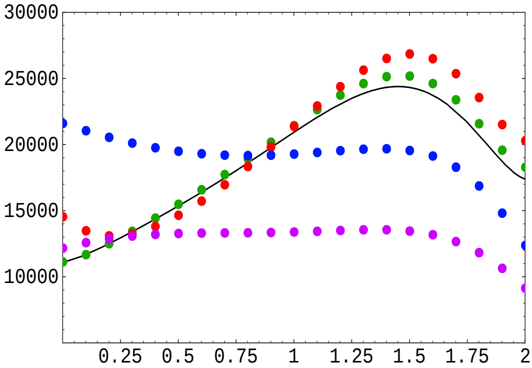
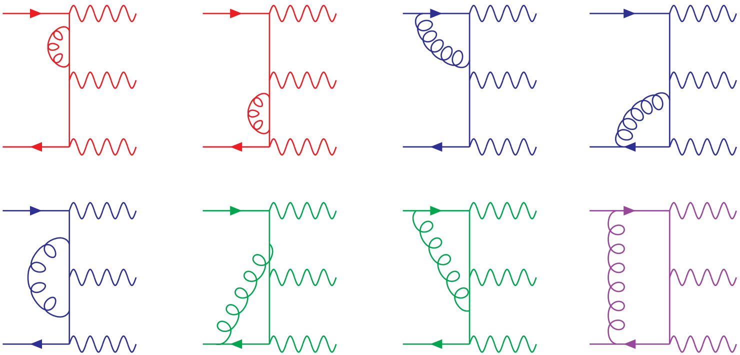


- Massless result [Mahlon]
- $m = 0.5$ GeV
- $m = 4.5$ GeV
- $m = 12.0$ GeV
- $m = 20.0$ GeV

$q\bar{q} \rightarrow ZZZ$ virtual corrections

A. Lazopoulos, K. Melnikov and F. Petriello, [arXiv:hep-ph/0703273]



Poles $1/\epsilon^2$ and $1/\epsilon$

$$\sigma^{\text{NLO,virt}}|_{\text{div}} = -C_F \frac{\alpha_s}{\pi} \frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}} (s_{12})^{-\epsilon} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) \sigma^{\text{LO}}$$

$q\bar{q} \longrightarrow ZZZ$ VIRTUAL CORRECTIONS

A still naive implementation

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- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices !

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Comparison with LMP

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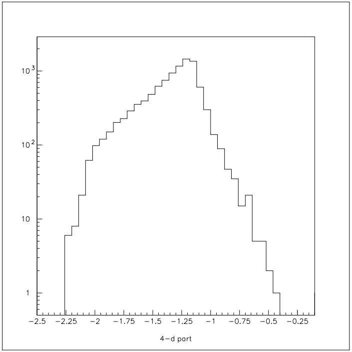
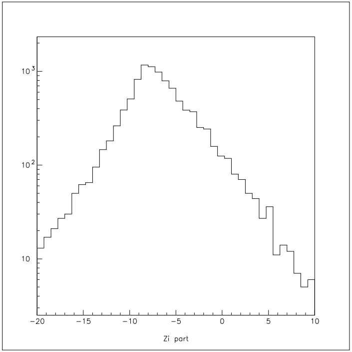
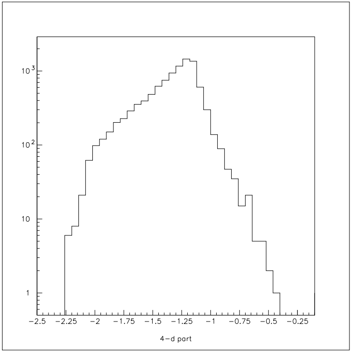
- An 'easy' agreement for all graphs with up to 4-point loop integrals
- A bit more work to uncover the differences in scalar function normalization that happen to show to order ϵ^2 thus influence only 5-point loop integrals.

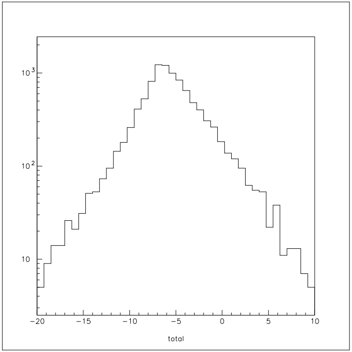
$q\bar{q} \rightarrow ZZZ$ VIRTUAL CORRECTIONS





$q\bar{q} \to ZZZ$ virtual corrections





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Typical time: 10^4 times faster (for non-singular PS-points)

$q\bar{q} \rightarrow ZZZ$ REAL CORRECTIONS

$$\sigma^{NLO}_{q\bar{q}} = \int\limits_{VVV} \left[d\sigma^{B}_{q\bar{q}} + d\sigma^{V}_{q\bar{q}} + d\sigma^{C}_{q\bar{q}} + \int\limits_{g} d\sigma^{A}_{q\bar{q}} \right] + \int\limits_{VVVg} \left[d\sigma^{R}_{q\bar{q}} - d\sigma^{A}_{q\bar{q}} \right]$$

$$\sigma^{NLO}_{gq} = \int\limits_{VVV} \left[+ d\sigma^{C}_{gq} \int\limits_{g} d\sigma^{A}_{gq} \right] + \int\limits_{VVVg} \left[d\sigma^{R}_{gq} - d\sigma^{A}_{gq} \right],$$

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$$\mathcal{D}^{q_1 g_6, \bar{q}_2} = \frac{8\pi\alpha_s C_F}{2\tilde{x}\, p_1 \cdot p_6} \left(\frac{1+\tilde{x}^2}{1-\tilde{x}} \right) |\mathcal{M}^B_{q\bar{q}}(\{\tilde{p}\})|^2$$

$$\tilde{x} = \frac{p_1 \cdot p_2 - p_2 \cdot p_6 - p_1 \cdot p_6}{p_1 \cdot p_2}$$

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$$d\sigma^R_{q\bar{q}} - d\sigma^A_{q\bar{q}} = \frac{1}{6}\frac{1}{N}\frac{1}{2s_{12}} \left[C_F\, |\mathcal{M}^R_{q\bar{q}}(\{p_j\})|^2 - \mathcal{D}^{q_1 g_6, \bar{q}_2} - \mathcal{D}^{\bar{q}_2 g_6, q_1} \right] d\Phi_{VVVg}$$

$q\bar{q} \to ZZZ$ REAL CORRECTIONS

$$d\sigma^C_{q\bar{q}} + \int_g d\sigma^A_{q\bar{q}} = \frac{\alpha_s C_F}{2\pi} \frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}} \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2\pi^2}{3}\right] d\sigma_B(\{p_j\})$$

$$+ \frac{\alpha_s C_F}{2\pi} \int_0^1 dx\, \mathcal{K}_{q\bar{q}}(x)\, d\sigma_B(xp_1, p_2; p_3, p_4, p_5)\; F_0(xp_1, p_2; p_3, p_4, p_5)$$

$$+ \frac{\alpha_s C_F}{2\pi} \int_0^1 dx\, \mathcal{K}_{q\bar{q}}(x)\, d\sigma_B(p_1, xp_2; p_3, p_4, p_5)\; F_0(p_1, xp_2; p_3, p_4, p_5)$$

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$$+ \frac{\alpha_s C_F}{2\pi} \int_0^1 dx\, \mathcal{K}_{q\bar{q}}(x)\, d\sigma_B(xp_1, p_2; p_3, p_4, p_5)\; F_0(xp_1, p_2; p_3, p_4, p_5)$$

$$+ \frac{\alpha_s C_F}{2\pi} \int_0^1 dx\, \mathcal{K}_{q\bar{q}}(x)\, d\sigma_B(p_1, xp_2; p_3, p_4, p_5)\; F_0(p_1, xp_2; p_3, p_4, p_5)$$

$$\mathcal{K}_{q\bar{q}}(x) = \left(\frac{1+x^2}{1-x}\right)_+ \log\left(\frac{s_{12}}{\mu_F^2}\right) + \left(\frac{4\log(1-x)}{1-x}\right)_+ + (1-x) - 2(1+x)\log(1-x)$$

$q\bar{q} \rightarrow ZZZ$ REAL CORRECTIONS

$$\sigma_{gq}^{NLO} = \int_{VVV} \left[\int_{g} d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int_{VVVg} \left[d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right]$$

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$$\sigma_{gq}^{NLO} = \int_{VVV} \left[\int_{g} d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int_{VVVg} \left[d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right]$$

$$d\sigma_{gq}^{R} - d\sigma_{gq}^{A} = \frac{1}{N}\frac{1}{2s_{12}} \left[T_R |\mathcal{M}_{gq}^{R}(\{p_j\}')|^2 F_1(\{p_j\}') - \mathcal{D}^{g_1 q_6, q_2} F_0(\{\tilde{p}_j\}) \right] d\Phi_{VVVq}$$

$q\bar{q} \to ZZZ$ REAL CORRECTIONS

$$\sigma_{gq}^{NLO} = \int_{VVV} \left[\int_g d\sigma_{gq}^A + d\sigma_{gq}^C \right] + \int_{VVVg} \left[d\sigma_{gq}^R - d\sigma_{gq}^A \right]$$

$$d\sigma_{gq}^R - d\sigma_{gq}^A = \frac{1}{N}\frac{1}{2s_{12}} \left[T_R |\mathcal{M}_{gq}^R(\{p_j\}')|^2 F_1(\{p_j\}') - \mathcal{D}^{g_1 q_6, q_2} F_0(\{\tilde{p}_j\}) \right] d\Phi_{VVVq}$$

$$\mathcal{D}^{g_1 q_6, q_2} = \frac{8\pi\alpha_s T_R}{\tilde{x}\, 2\, p_1 \cdot p_6} [1 - 2\,\tilde{x}\,(1-\tilde{x})] |\mathcal{M}_{q\bar{q}}^B(\{\tilde{p}_j\})|^2$$

$q\bar{q} \to ZZZ$ REAL CORRECTIONS

$$d\sigma_{gq}^{C} + \int_{\bar{q}} d\sigma_{gq}^{A} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx\, \mathcal{K}_{gq}(x)\, d\sigma_B(xp_1, p_2; p_3, p_4, p_5)\; F_0(xp_1, p_2; p_3, p_4, p_5)$$

$$\mathcal{K}_{gq}(x) = [x^2 + (1-x)^2] \log\left(\frac{s_{12}}{\mu_F^2}\right) + 2x(1-x) + 2[x^2 + (1-x)^2] \log(1-x)$$

$q\bar{q} \to ZZZ$ REAL CORRECTIONS

$$d\sigma_{gq}^{C} + \int_{\bar{q}} d\sigma_{gq}^{A} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx\, \mathcal{K}_{gq}(x)\, d\sigma_B(xp_1, p_2; p_3, p_4, p_5)\; F_0(xp_1, p_2; p_3, p_4, p_5)$$

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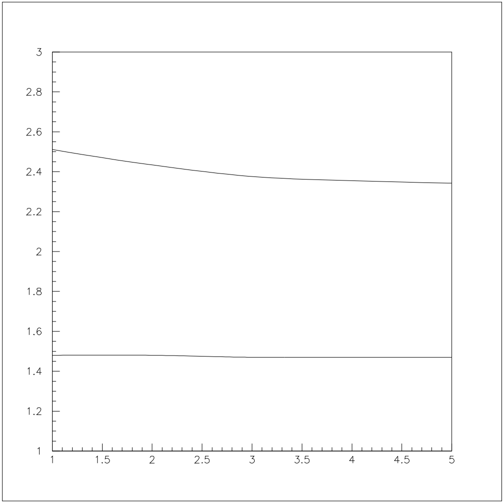
- check also with phase-space slicing method

$q\bar{q} \rightarrow ZZZ$ NLO

scale	σ_0	σ_V/σ_0	σ_R	σ_{NLO}
$\mu = M_Z$	1.481(5)	0.536(1)	0.238(2)	2.512(2)
$\mu = 2M_Z$	1.487(5)	0.481(1)	0.232(2)	2.434(2)
$\mu = 3M_Z$	1.477(5)	0.452(1)	0.232(2)	2.376(2)
$\mu = 4M_Z$	1.479(5)	0.436(1)	0.232(2)	2.355(2)
$\mu = 5M_Z$	1.479(5)	0.424(1)	0.237(2)	2.343(2)

$q\bar{q} \to ZZZ$ NLO



Reduction at the integrand level

Outlook

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- changes the computational approach at one loop

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- Numerical but still algebraic: speed and precision not a problem

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- Understand potential stability problems

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- Understand potential stability problems
- Combine with the real corrections

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- Understand potential stability problems
- Combine with the real corrections
- Automatize through Dyson-Schwinger equations

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A generic NLO calculator seems feasible

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`CUTTOOLS version 0.` is ready !