Reduction of one-loop amplitudes at the integrand level

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Outline

Introduction: LHC needs NLO

- The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course)

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- The current need of precision goes beyond tree order. At LHC, most analyses require at least next-to-leading order calculations (NLO)
- As a result, a big effort has been devoted by several groups to the problem of an efficient computation of one-loop corrections for multi-particle processes!

[from G. Heinrich's Summary talk]

Wishlist Les Houches 2007
1. $pp \rightarrow V\,V + \text{jet}$
2. $pp \rightarrow t\bar{t}\,b\bar{b}$
3. $pp \rightarrow t\bar{t} + 2\ \text{jets}$
4. $pp \rightarrow W\,W\,W$
5. $pp \rightarrow V\,V\,b\bar{b}$
6. $pp \rightarrow V\,V + 2\ \text{jets}$
7. $pp \rightarrow V + 3\ \text{jets}$
8. $pp \rightarrow t\bar{t}\,b\bar{b}$
9. $pp \rightarrow 4\ \text{jets}$

Processes for which a NLO calculation is both desired and feasible

Will we "finish" in time for LHC?

What has been done? (2005-2007)

Some recent results → Cross Sections available

- $pp \to Z\,Z\,Z$ $pp \to t\bar{t}Z$ [Lazopoulos, Melnikov, Petriello]
- $pp \to H + 2$ jets [Campbell, et al., J. R. Andersen, et al.]
- $pp \to VV + 2$ jets via VBF [Bozzi, Jäger, Oleari, Zeppenfeld]

Mostly 2 → 3, very few 2 → 4 complete calculations.

- $e^+\ e^- \to 4$ fermions [Denner, Dittmaier, Roth]
- $e^+\ e^- \to H\,H\,\nu\,\bar{\nu}$ [GRACE group (Boudjema et al.)]

This is NOT a complete list

(A lot of work has been done at NLO → calculations & new methods)

NLO TROUBLES

Problems arising in NLO calculations

- Large Number of Feynman diagrams
- Reduction to Scalar Integrals (or sets of known integrals)
- Numerical Instabilities (inverse Gram determinants, spurious phase-space singularities)
- Extraction of soft and collinear singularities (we need virtual and real corrections)

Methods available

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
  - general applicability major achievements
  - but major problem: not designed @ amplitude level

Methods available

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
- Semi-Numerical Approach (Algebraic/Partly Numerical – Improved traditional) $\rightarrow$ Reduction to set of well-known integrals
- Numerical Approach (Numerical/Partly Algebraic) $\rightarrow$ Compute tensor integrals numerically
  - Ellis, Giele, Glover, Zanderighi;
  - Binoth, Guillet, Heinrich, Schubert;
  - Denner, Dittmaier; Del Aguila, Pittau;
  - Ferroglia, Passera, Passarino, Uccirati;
  - Nagy, Soper; van Hameren, Vollinga, Weinzierl;

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- Analytic Approach (Twistor-inspired)
  $\rightarrow$ extract information from lower-loop, lower-point amplitudes
  $\rightarrow$ determine scattering amplitudes by their poles and cuts
  - major advantage: designed to work @ amplitude level
  - **quadruple and triple cuts major simplifications**
  - Bern, Dixon, Dunbar, Kosower, Berger, Forde;
  - Anastasiou, Britto, Cachazo, Feng, Kunszt, Mastrolia;

Methods available

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⋆ OPP Integrand-level reduction **combine: PV@integrand + n-particle cuts**

G. Ossola., C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007) – arXiv:hep-ph/0609007

and JHEP **0707** (2007) 085 – arXiv:0704.1271 [hep-ph]

Any m-point one-loop amplitude can be written, before integration, as

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

A bar denotes objects living in $n = 4 + \epsilon$ dimensions

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2$$

$$\bar{q}^2 = q^2 + \tilde{q}^2$$

$$\bar{D}_i = D_i + \tilde{q}^2$$

External momenta p_i are 4-dimensional objects

The old "master" formula

$$\begin{aligned}
\int A &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2) \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1) \\
&+ \sum_{i_0}^{m-1} a(i_0) A_0(i_0) \\
&+ \text{rational terms}
\end{aligned}$$

OPP "MASTER" FORMULA - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

OPP "MASTER" FORMULA - II

$$\begin{aligned} N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) \qquad\qquad \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c(i_0 i_1 i_2) \qquad\qquad] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\ &+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) \qquad\qquad \right] \prod_{i \neq i_0, i_1}^{m-1} D_i + \sum_{i_0}^{m-1} [a(i_0) \qquad\qquad] \prod_{i \neq i_0}^{m-1} D_i \end{aligned}$$

- The quantities $d(i_0 i_1 i_2 i_3)$ are the coefficients of 4-point functions with denominators labeled by i_0, i_1, i_2, and i_3.
- $c(i_0 i_1 i_2)$, $b(i_0 i_1)$, $a(i_0)$ are the coefficients of all possible 3-point, 2-point and 1-point functions, respectively.

OPP "MASTER" FORMULA - II

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

The quantities $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ are the "spurious" terms

- They still depend on q (integration momentum)
- They should vanish upon integration

What is the explicit expression of the spurious term?

Spurious Terms - I

Following F. del Aguila and R. Pittau, arXiv:hep-ph/0404120

- Express any q in $N(q)$ as

$$q^\mu = -p_0^\mu + \sum_{i=1}^4 G_i\, \ell_i^\mu \;,\;\; {\ell_i}^2 = 0$$

$$k_1 = \ell_1 + \alpha_1 \ell_2 \,, \quad k_2 = \ell_2 + \alpha_2 \ell_1 \,,\;\; k_i = p_i - p_0$$
$${\ell_3}^\mu =< \ell_1|\gamma^\mu|\ell_2] \,,\;\; {\ell_4}^\mu =< \ell_2|\gamma^\mu|\ell_1]$$

- The coefficients G_i either reconstruct denominators D_i

$\rightarrow$ They give rise to d, c, b, a coefficients

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- The coefficients G_i either reconstruct denominators D_i or vanish upon integration

$\rightarrow$ They give rise to d, c, b, a coefficients
$\rightarrow$ They form the spurious $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ coefficients

- $\tilde{d}(\mathrm{q})$ term (only 1)

$$\tilde{d}(q) = \tilde{d}\, T(q)\,,$$

where $\tilde{d}$ is a constant (does not depend on q)

$$T(q) \equiv Tr[(\not{q} + \not{p}_0)\not{\ell}_1\not{\ell}_2\not{k}_3\gamma_5]$$

Spurious Terms - II

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- $\tilde{c}(\mathrm{q})$ terms (they are 6)

$$\tilde{c}(q) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j}[(q+p_0)\cdot\ell_3]^j + \tilde{c}_{2j}[(q+p_0)\cdot\ell_4]^j \right\}$$

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- $\tilde{b}(\mathrm{q})$ and $\tilde{a}(\mathrm{q})$ give rise to 8 and 4 terms, respectively

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

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- Melrose, Nuovo Cim. 40 (1965) 181
- G. Källén, J.Toll, J. Math. Phys. 6, 299 (1965)

General strategy

Now we know the form of the spurious terms:

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
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\end{aligned}
$$

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Extract all the coefficients by evaluating N(q) for a set of values of the integration momentum q

There is a very good set of such points: Use values of q for which a set of denominators D_i vanish → The system becomes "triangular": solve first for 4-point functions, then 3-point functions and so on

Example: 4-particles process

$$
\begin{aligned}
N(q) &= d + \tilde{d}(q) + \sum_{i=0}^{3} [c(i) + \tilde{c}(q;i)]\, D_i + \sum_{i_0<i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q;i_0 i_1)\right] D_{i_0} D_{i_1} \\
&+ \sum_{i_0=0}^{3} [a(i_0) + \tilde{a}(q;i_0)]\, D_{i\neq i_0} D_{j\neq i_0} D_{k\neq i_0}
\end{aligned}
$$

We look for a q of the form $q^\mu = -p_0^\mu + x_i \ell_i^\mu$ such that

$$D_0 = D_1 = D_2 = D_3 = 0$$

$\rightarrow$ we get a system of equations in x_i that has two solutions $q_0^\pm$

Example: 4-particles process

$$N(q) = d + \tilde{d}(q)$$

Our "master formula" for $q = q_0^{\pm}$ is:

$$N(q_0^{\pm}) = [d + \tilde{d}\ T(q_0^{\pm})]$$

$\longrightarrow$ solve to extract the coefficients d and $\tilde{d}$

Example: 4-particles process

$$N(q) - d - \tilde{d}(q) = \sum_{i=0}^{3} \left[c(i) + \tilde{c}(q;i)\right] D_i + \sum_{i_0<i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] D_{i_0} D_{i_1}$$
$$+ \sum_{i_0=0}^{3} \left[a(i_0) + \tilde{a}(q; i_0)\right] D_{i\neq i_0} D_{j\neq i_0} D_{k\neq i_0}$$

Then we can move to the extraction of c coefficients using

$$N'(q) = N(q) - d - \tilde{d}\, T(q)$$

and setting to zero three denominators (ex: $D_1 = 0$, $D_2 = 0$, $D_3 = 0$)

Example: 4-particles process

$$N(q) - d - \tilde{d}(q) \quad = \quad [c(0) + \tilde{c}(q;0)]\, D_0$$

We have infinite values of q for which

$$D_1 = D_2 = D_3 = 0 \quad \text{and} \quad D_0 \neq 0$$

$\rightarrow$ Here we need 7 of them to determine $c(0)$ and $\tilde{c}(q;0)$

- Let's go back to the integrand

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

- Insert the expression for $N(q) \rightarrow$ we know all the coefficients

$$N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d + \tilde{d}(q)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c + \tilde{c}(q)] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i + \cdots$$

- Finally rewrite all denominators using

$$\frac{D_i}{\bar{D}_i} = \bar{Z}_i \,, \qquad \text{with} \quad \bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)$$

Rational Terms - II

$$
\begin{aligned}
A(\bar{q}) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \prod_{i \neq i_0, i_1}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \prod_{i \neq i_0}^{m-1} \bar{Z}_i
\end{aligned}
$$

The rational part is produced, after integrating over $d^n q$, by the $\tilde{q}^2$ dependence in $\bar{Z}_i$

$$
\bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)
$$

Rational Terms - III

The "Extra Integrals" are of the form

$$I_{s;\mu_1\cdots\mu_r}^{(n;2\ell)} \equiv \int d^n q\, \tilde{q}^{2\ell} \frac{q_{\mu_1}\cdots q_{\mu_r}}{\bar{D}(k_0)\cdots\bar{D}(k_s)}\,,$$

where

$$\bar{D}(k_i) \equiv (\bar{q}+k_i)^2 - m_i^2\,,\, k_i = p_i - p_0$$

These integrals:

- **have dimensionality** $\mathcal{D} = 2(1+\ell-s)+r$
- **contribute only when** $\mathcal{D} \geq 0$**, otherwise are of** $\mathcal{O}(\epsilon)$

SUMMARY

Calculate $N(q)$

- We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly
- Calculate $N(q)$ numerically via recursion relations
- Just specify external momenta, polarization vectors and masses and proceed with the reduction!

Compute all coefficients

- by evaluating N(q) at certain values of integration momentum

Evaluate scalar integrals

- massive integrals $\longrightarrow$ FF [G. J. van Oldenborgh]
- massless integrals $\longrightarrow$ OneLOop [A. van Hameren]

What we gain

- PV:

- Unitarity Methods:

What we gain

- PV:
  - $N(q)$ or $A(q)$ hasn't to be known analytically

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  - A solid way to get all rational terms

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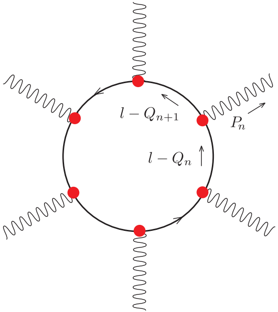
- Different ways of solving it, besides 'unitarity method'

The Master Equation

The $N \equiv N$ test

A tool to efficiently treat phase-space points with numerical instabilities

As an example we present 4-photon and 6-photon amplitudes (via fermionic loop of mass m_f)



Input parameters for the reduction:

- External momenta p_i
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- Polarization vectors

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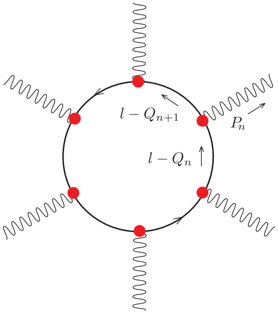

Input parameters for the reduction:

- External momenta $p_i \longrightarrow$ in this example massless, i.e. $p_i^2 = 0$
- Masses of propagators in the loop $\longrightarrow$ all equal to m_f
- Polarization vectors $\longrightarrow$ various helicity configurations

Four Photons – Comparison with *Gounaris et al.*

$$\frac{F^f_{++++}}{\alpha^2 Q_f^4} = -8$$

Rational Part

$$
\begin{aligned}
\frac{F^{f}_{++++}}{\alpha^2 Q_f^4} &= -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right)B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right)B_0(\hat{t}) \\
&\quad - 8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}\qquad\qquad\right)\left[\hat{t}\,C_0(\hat{t}) + \hat{u}\,C_0(\hat{u})\right] \\
&\quad - 4\left[\qquad\qquad\qquad \frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}\qquad\qquad\right]D_0(\hat{t},\hat{u})
\end{aligned}
$$

Massless four-photon amplitudes

$$
\begin{aligned}
\frac{F^f_{++++}}{\alpha^2 Q_f^4} &= -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right)B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right)B_0(\hat{t}) \\
&- 8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} - \frac{4m_f^2}{\hat{s}}\right)[\hat{t}C_0(\hat{t}) + \hat{u}C_0(\hat{u})] \\
&- 4\left[4m_f^4 - (2\hat{s}m_f^2 + \hat{t}\hat{u})\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} + \frac{4m_f^2\hat{t}\hat{u}}{\hat{s}}\right]D_0(\hat{t},\hat{u}) \\
&+ 8m_f^2(\hat{s}-2m_f^2)[D_0(\hat{s},\hat{t}) + D_0(\hat{s},\hat{u})]
\end{aligned}
$$

Massive four-photon amplitudes

Four Photons – Comparison with *Gounaris et al.*

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\begin{aligned}
\frac{F^f_{++++}}{\alpha^2 Q_f^4} \quad = \quad & -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right) B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right) B_0(\hat{t}) \\
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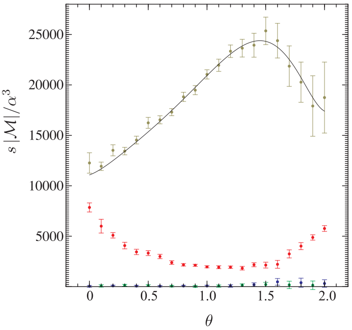
\end{aligned}
$$

Massive four-photon amplitudes

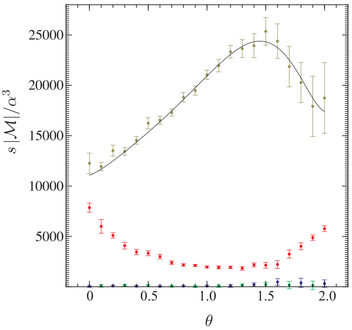
Results also checked for F^f_{+++-} and F^f_{++--}

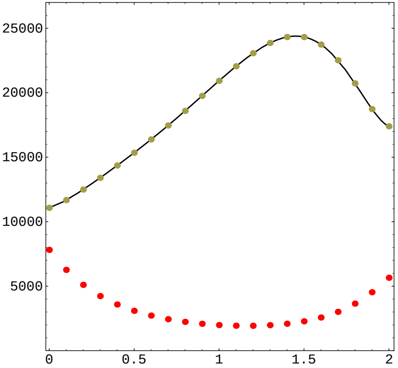
Massless case: $[++----]$ and $[+--++-]$



Plot presented by Nagy and Soper hep-ph/0610028
(also Binoth et al., hep-ph/0703311)

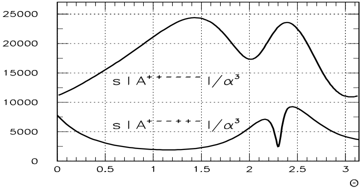
Massless case: $[+ + - - - -]$ and $[+ - - + + -]$

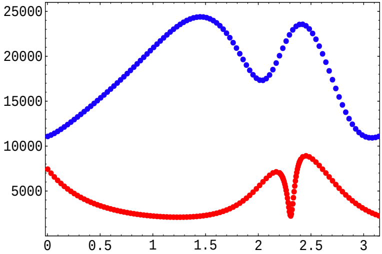




Analogous plot produced with OPP reduction

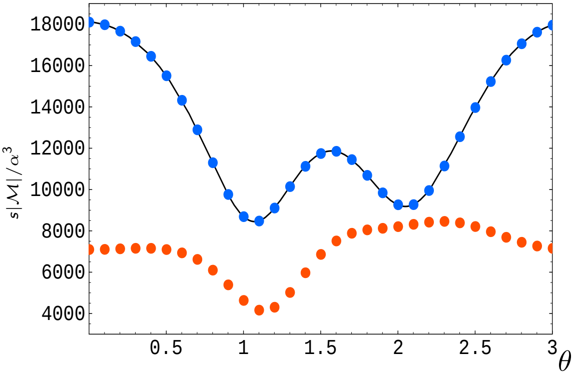
Massless case: $[++----]$ and $[++--+-]$





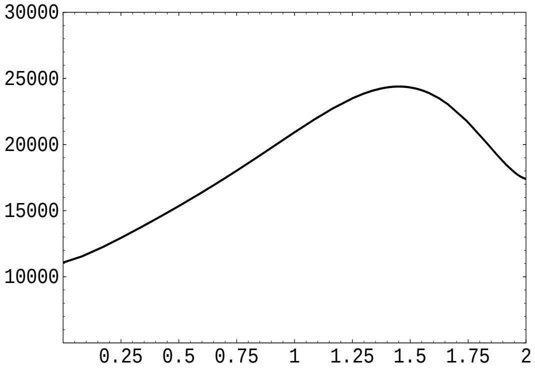
Same plot as before for a wider range of θ

Massless case: $[++----\,-]$ and $[++--+-]$



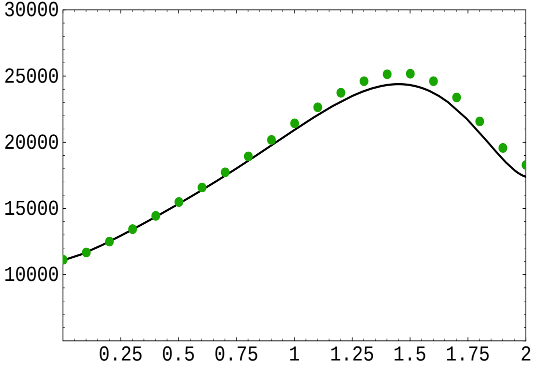
Same idea for a different set of external momenta

Six Photons with Massive Fermions



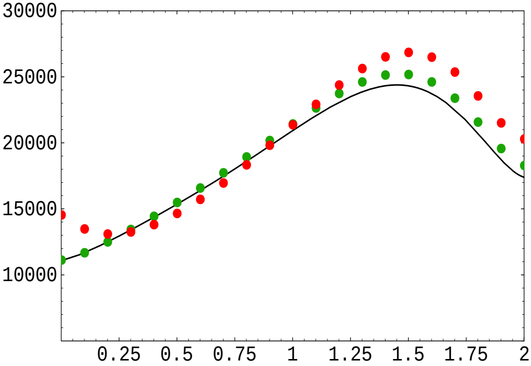
- Massless result [Mahlon]

Six Photons with Massive Fermions

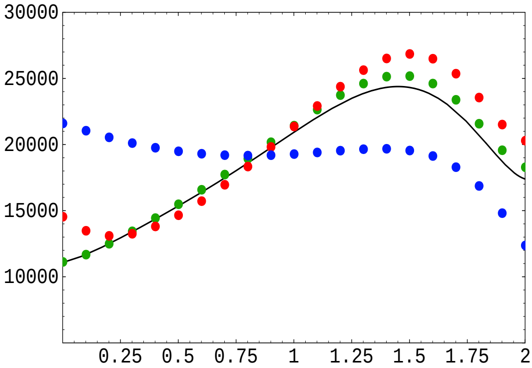


- Massless result [Mahlon]
- $m = 0.5$ GeV

Six Photons with Massive Fermions



- Massless result [Mahlon]
- $m = 0.5$ GeV
- $m = 4.5$ GeV



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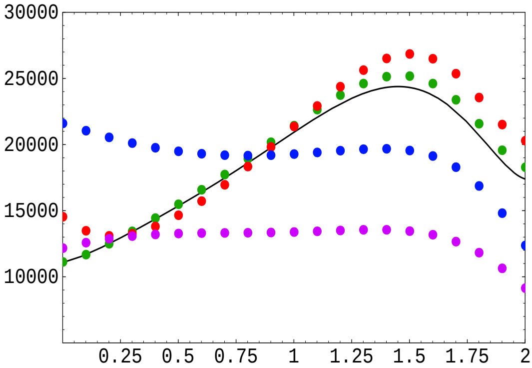
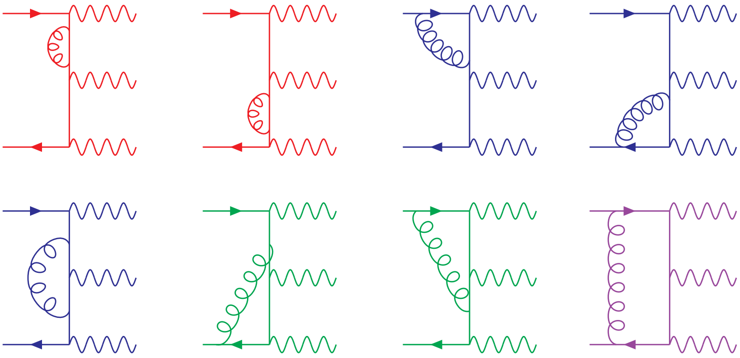
SIX PHOTONS WITH MASSIVE FERMIONS



- Massless result [Mahlon]
- $m = 0.5$ GeV
- $m = 4.5$ GeV
- $m = 12.0$ GeV
- $m = 20.0$ GeV

$q\bar{q} \to ZZZ$ VIRTUAL CORRECTIONS

A. Lazopoulos, K. Melnikov and F. Petriello, [arXiv:hep-ph/0703273]



POLES $1/\epsilon^2$ AND $1/\epsilon$

$$\sigma^{\mathrm{NLO,virt}}|_{\mathrm{div}} = -C_F \frac{\alpha_s}{\pi} \frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}} (s_{12})^{-\epsilon} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) \sigma^{\mathrm{LO}}$$

$q\bar{q} \to ZZZ$ VIRTUAL CORRECTIONS

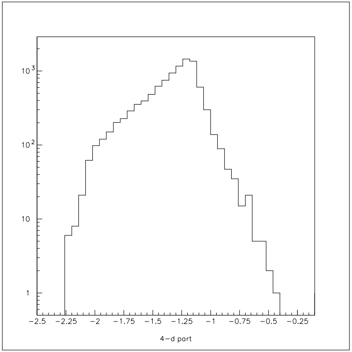
A very naive implementation

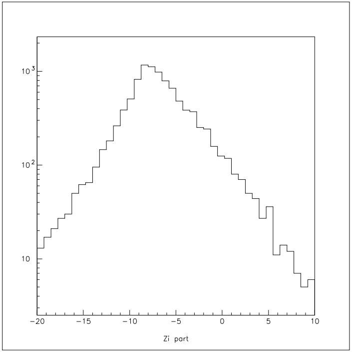
- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices !
- Calculate 4d and rational terms graph by graph

Comparison with LMP

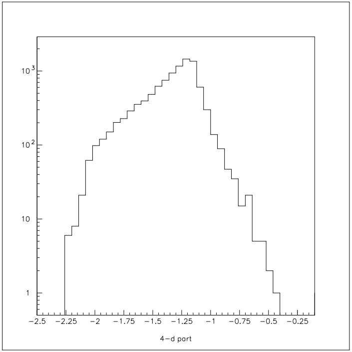
- Of course full agreement for the $1/\epsilon^2$ and $1/\epsilon$ terms
- An 'easy' agreement for all graphs with up to 4-point loop integrals
- A bit more work to uncover the differences in scalar function normalization that happen to show to order ϵ^2 thus influence only 5-point loop integrals.

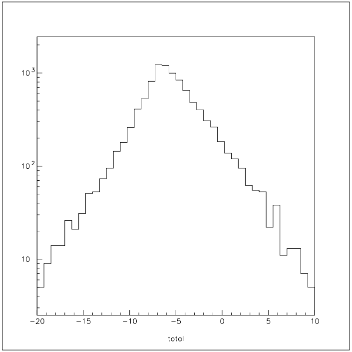
$q\bar{q} \to ZZZ$ virtual corrections





$q\bar{q} \rightarrow ZZZ$ VIRTUAL CORRECTIONS





$q\bar{q} \longrightarrow ZZZ$ VIRTUAL CORRECTIONS

Typical precision:

- LMP: 9.573(66)
- OPP:

$$\begin{cases} -26.4570674281\,5552 \\ -26.4570674281\,6550366101855793772342 6 \end{cases}$$

Typical time: 200 times faster

Outlook

Reduction at the integrand level

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- changes the computational approach at one loop

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`CUTTOOLS version 0.` is ready !

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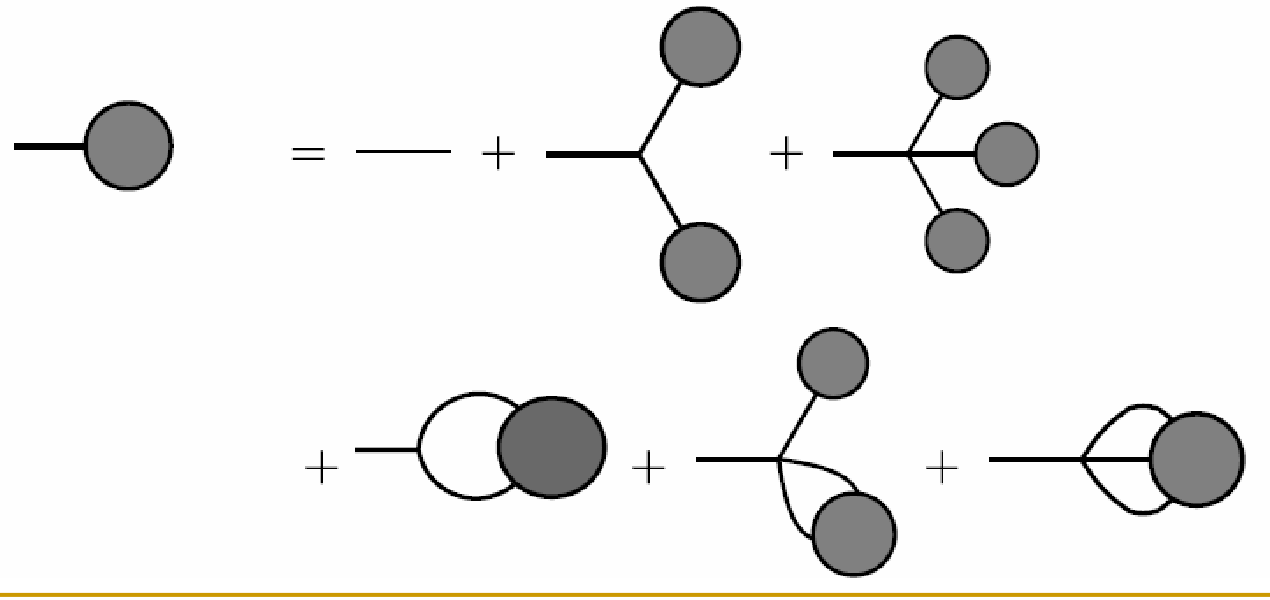
- The DS equations to all orders
  - Imagine a theory with 3- and 4- point vertices and just one field. Then it is straightforward to write an equation that gives the amplitude for $1 \longrightarrow n$



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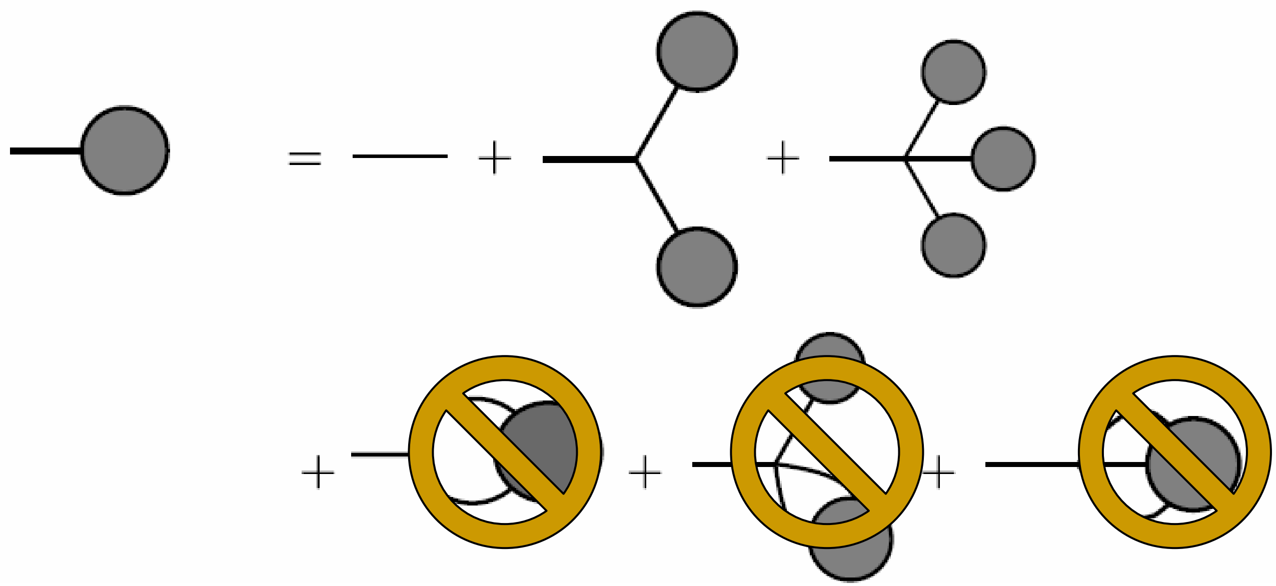

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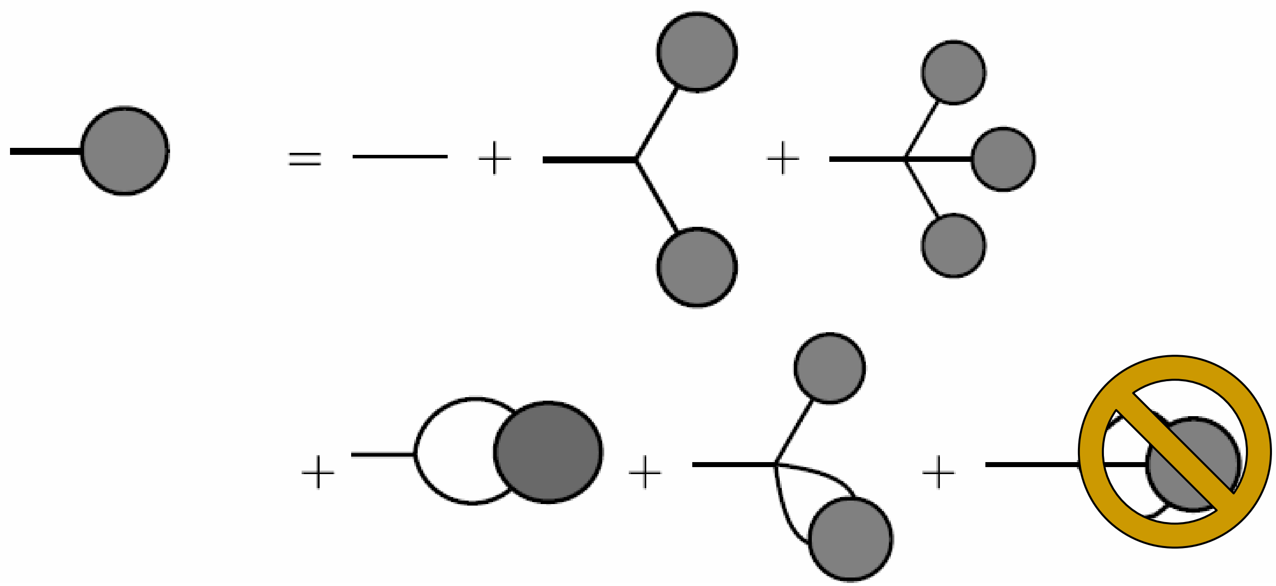
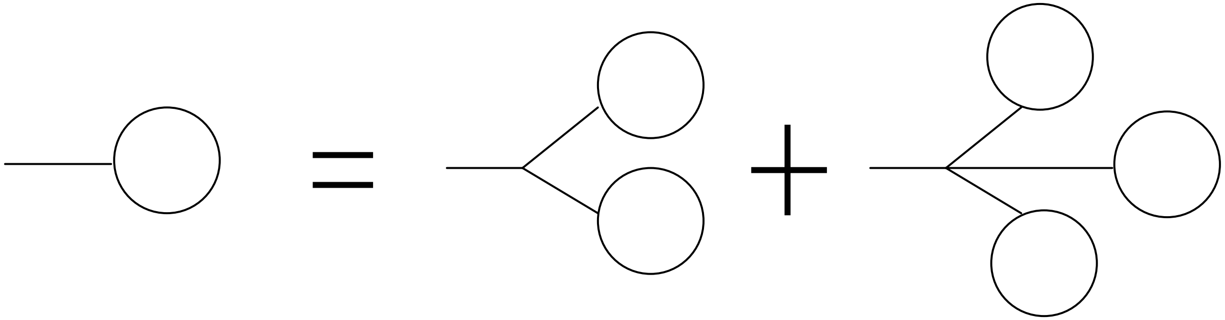


The Outlook

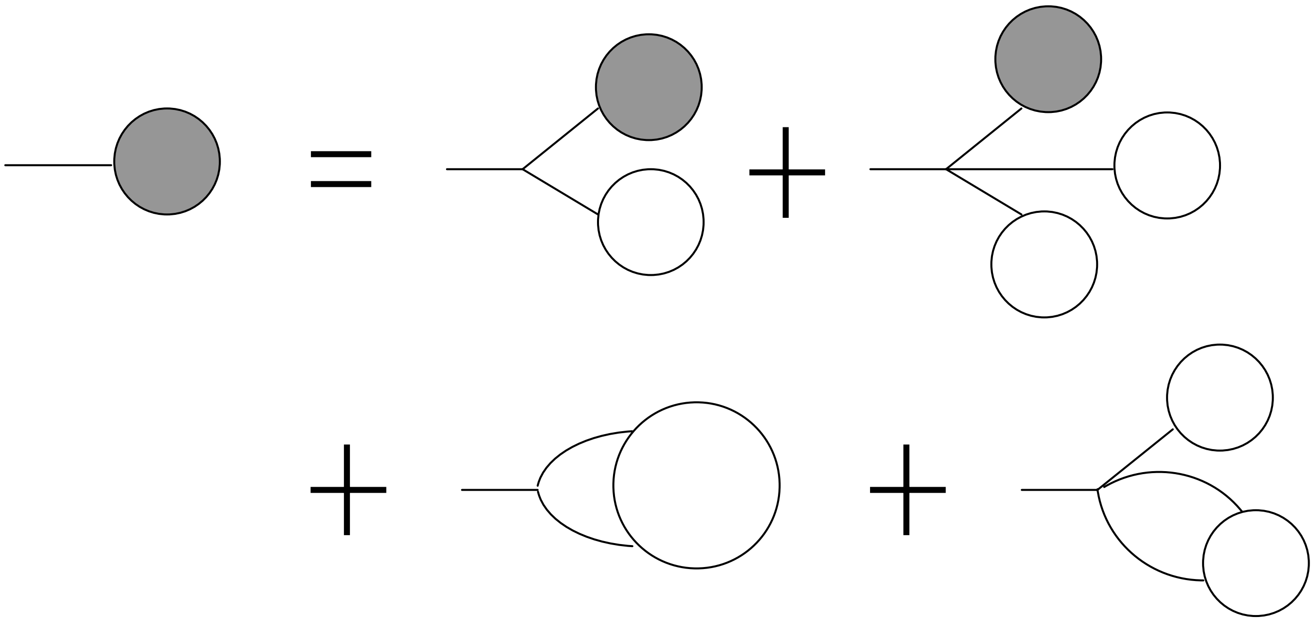
- The DS equations to tree order



- Combine 2,3,….,n external particles

The Outlook

- The DS equations to one loop order linear !



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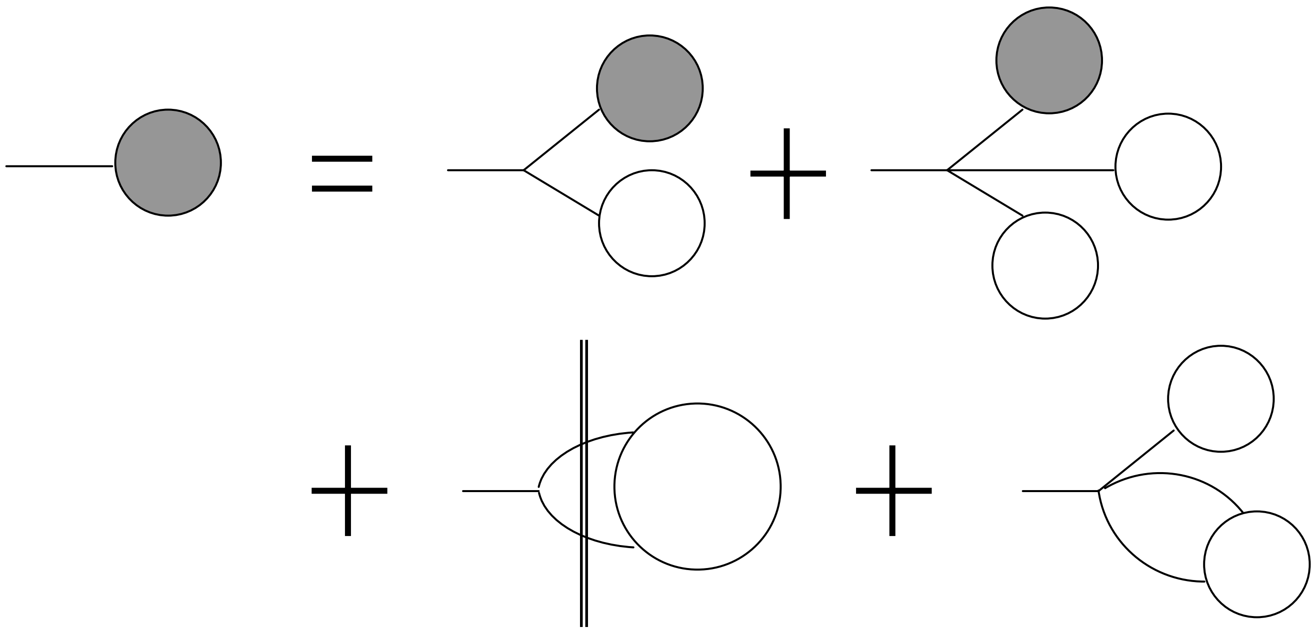

  - N+1 tree order sub-amplitudes